6.341 Project II: Spectral Estimation

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A.1 Periodogram-based Estimation Techniques

Methodology:

Periodogram

The Periodogram could be obtained by

\[
P_{\text{per}}(e^{j\omega}) = \text{DTFT}\{R_{xx}[m]\} = \frac{1}{N} \left| X_N(e^{j\omega}) \right|^2
\]

\[
R_{xx}[m] = \frac{1}{N} \sum_{n=0}^{N-1-k} x[n + k]x^*[n]
\]

Estimated autocorrelation of \(x[n]\) with window length = \(N\)

\[
X_N(e^{j\omega})
\]

DTFT of \(x[n]\) with window length = \(N\)

In this particular problem, we have \(N = 2048\)

When computing periodogram, we can either choose to compute \(R_{xx}[m]\) first and then do the DTFT, or to compute DTFT of \(x[n]\) first and then do the point-wise multiplication with its conjugate. Considering the computing complexity and the implementation convenience, I choose to compute DTFT of \(x[n]\) first.

There is something we should care about. We cannot do DTFT in computer; the only thing we can do is DFT. In order to prevent time-domain aliasing, we have to choose \(N\) larger than or equals to 2048. Here I fix \(N\) to be 2048 through this project. The resulting periodogram will have 2048 sample points. We only need the first half of the samples to represent the one-sided periodogram. The odd samples (1, 3, 5, 7...) are the frequencies that the given true power spectrum is sampled at.

The overall function does the followings:

1. Take a 2048-point FFT of input sequence \(x[n]\) (Given that the samples in \(x[n]\) will not exceed 2048).
2. Multiply the result with its conjugate and then divide it by the length of input sequence \(x[n]\).
**Welch’s Method**
The function that implementing the Welch’s method takes two input parameters: the number of segments and the overlap percentage. Given these two parameters, I can compute the maximum sequence length. The expression of the maximum sequence length is given by

\[
L \times S \leq N + L \times O \times (S - 1)
\]

\[
L_{\text{max}} = \text{floor}\left[\frac{N}{S - O(S - 1)}\right]
\]

L is the length of a segment. S is the number of the segments. O is the overlap ratio. N is the number of the samples in the input (2048). We want L to be as large as possible because we generally desire high resolution.

Once we have the length of a segment, we can divide the input sequence \(x[n]\) into S subsequences. Then, we feed these subsequences into our periodogram function, and we will get S different periodograms. At last, we average all the periodograms to get the result.

**Bartlett’s Method**
Bartlett’s method is just a special case of Welch’s method with input parameter overlap percentage equals to zero.

**Result:**
V1
<table>
<thead>
<tr>
<th>Method</th>
<th>Means Square Error J</th>
</tr>
</thead>
<tbody>
<tr>
<td>Periodogram</td>
<td>16.527</td>
</tr>
<tr>
<td>Bartlett’s method (S=4, O=0.0)</td>
<td>4.0852</td>
</tr>
<tr>
<td>Bartlett’s method (S=16, O=0.0)</td>
<td>1.0441</td>
</tr>
<tr>
<td>Welch’s method (S=4, O=0.2)</td>
<td>4.7524</td>
</tr>
<tr>
<td>Welch’s method (S=4, O=0.4)</td>
<td>5.1771</td>
</tr>
<tr>
<td>Welch’s method (S=4, O=0.6)</td>
<td>6.7094</td>
</tr>
<tr>
<td>Welch’s method (S=4, O=0.8)</td>
<td>9.9949</td>
</tr>
</tbody>
</table>
V2

Periodogram

Amplitude

Frequency

Bartlett's method with 4 segments

$J = 36.121$

Truth

Bartlett's method with 16 segments

$J = 25.365$

Truth

Welch's method with 50% overlap

$J = 34.503$

Truth

Welch's method with 40% overlap

$J = 41.505$

Truth
<table>
<thead>
<tr>
<th>Method</th>
<th>Means Square Error J</th>
</tr>
</thead>
<tbody>
<tr>
<td>Periodogram</td>
<td>56.121</td>
</tr>
<tr>
<td>Bartlett’s method (S=4, O=0.0)</td>
<td>26.347</td>
</tr>
<tr>
<td>Bartlett’s method (S=16, O=0.0)</td>
<td>25.305</td>
</tr>
<tr>
<td>Welch’s method (S=4, O=0.2)</td>
<td>34.993</td>
</tr>
<tr>
<td>Welch’s method (S=4, O=0.4)</td>
<td>41.935</td>
</tr>
<tr>
<td>Welch’s method (S=4, O=0.6)</td>
<td>48.822</td>
</tr>
<tr>
<td>Welch’s method (S=4, O=0.8)</td>
<td>59.668</td>
</tr>
</tbody>
</table>

Number of Segments

Given the data above, we can observe that the increase in the number of segments will reduce J. This is expected because J is proportional to the variance of the estimate spectrums. When the number of the segments increase, the variance of the estimate spectrums decrease; therefore, J decrease. However, in the Bartlett’s method, a significant decrease in J happens in V1 when S goes from 4 to 16, but only a slight decrease in J happens in V2. The reason may be that the peak values in V2 are drawn down because of the averaging effect, which will increase J. Therefore, the two effects cancel each other and yield a smaller amount of decrease in J.

The following plots are the experiment data of different numbers of segments:

- Jmin = 0.0978
  - $J^{\star} = 256$

- Jmin = 22.7861
  - $J^{\star} = 8$
According to the plots above, we learn that for the smooth spectrum like P1, we can set the number of segments to be very large to get better J because we care only about variance but not resolution. However, for the peaky spectrum like P2, we cannot set the number too high because we need longer segment for higher resolution.

Overlap Percentage
When comparing different amount of overlap percentage of the Welch’s method, we find that the increase in overlap will increase J. If we take each segment as different realizations of a random process, higher overlap means longer segments, so we should get the periodogram with higher resolution and about the same variance. However, the segments are not different realizations. When the overlap goes high, they are close related with each other. Therefore, we end up averaging very similar periodograms, which cannot reduce the variance. In a word, increase in overlap will increase the resolution and the variance. Generally, we care about both resolution and variance so we will get an optimal overlap percentage (~50%). However, in this problem, the only criterion is J. So we expect that the increase in overlap will worsen J. The following plots are the experiment data of different overlap percentage:

A.2 Auto regressive Spectrum Estimation

Methodology:
Use the built-in function pyulear in Matlab.
1. The default number of the samples is 128. We should set the number of the samples to 512 so that we can compare it to the true spectrum easily.
2. There is default π attenuation applying to the spectrum. We should compensate the effect by multiplying every sample by π.
Result:

V1
<table>
<thead>
<tr>
<th>Order</th>
<th>Means Square Error J</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.8598</td>
</tr>
<tr>
<td>2</td>
<td>0.6196</td>
</tr>
<tr>
<td>3</td>
<td>1.8105</td>
</tr>
<tr>
<td>4</td>
<td>0.7267</td>
</tr>
<tr>
<td>5</td>
<td>0.3674</td>
</tr>
<tr>
<td>6</td>
<td>0.3481</td>
</tr>
<tr>
<td>7</td>
<td>0.6781</td>
</tr>
<tr>
<td>8</td>
<td>0.4793</td>
</tr>
</tbody>
</table>

V2
Order Means Square Error $J$

<table>
<thead>
<tr>
<th>Order</th>
<th>$J$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>133.15</td>
</tr>
<tr>
<td>2</td>
<td>120.17</td>
</tr>
<tr>
<td>3</td>
<td>126.97</td>
</tr>
<tr>
<td>4</td>
<td>130.36</td>
</tr>
<tr>
<td>5</td>
<td>129.35</td>
</tr>
<tr>
<td>6</td>
<td>119.57</td>
</tr>
<tr>
<td>7</td>
<td>78.37</td>
</tr>
<tr>
<td>8</td>
<td>46.46</td>
</tr>
</tbody>
</table>

Order
We can think of autoregressive estimation as polynomial approximation. We want to find a polynomial $(1 + A(z^{-1}))$ with order $N$ so that

$$\frac{1}{1 + A(z^{-1})} = \phi(z)$$

We know that in order to get reasonable approximation, the order should be larger than the number of extremals + 1. According to the true spectrums of $v_1$ and $v_2$, we know that the order of $v_1$ should be larger than 3 (2 extremals) and the order of $v_2$ should be larger than 6 (5 extremals). The significant decrease in $J$ of $v_2$ when the order goes from 6 to 7 gives an example.

In addition, the higher order is, the closer the approximation will be. We can find this trend in $v_2$. However, the minimum of $J$ happens when order equals to 6 for $v_1$ and then $J$ starts to go up when order increase. The reason may be that we are using $R_{xx}[m]$ to obtain the coefficient of $A$ and $R_{xx}[m]$ is just estimated autocorrelation, not true autocorrelation. For low order signal like $v_1$, if $R_{xx}[m]$ is true autocorrelation, we will get closer approximation when using higher order. However, if we get only finite samples, the approximation could not be closer because of the
error of the estimated autocorrelation $R_{xx}[m]$. Instead, we may introduce false information and worsen the approximation.

In the lecture, we have derived that

$$E[R_{xx}[m]] = \frac{N - |m|}{N} R_{\text{real}}[m] \text{ for } |m| < N$$

That means that $R_{xx}[m]$ is not reliable for large $m$.

Since we are using Yule-Walker Equation to obtain the coefficients of $A$, when the order is high, we have to use $R_{xx}[m]$ with high $m$. Therefore, we can conclude that the autoregressive spectrum estimation is not reliable for large order.

The following plots are the experiment data of different orders. We can observe that $J$ will increase with order when the order is high.

![Experiment data plots](image)

**A.3 Reflection**

Autoregressive spectrum estimation works better for signal $v_1$ and periodogram-based estimation works better for signal $v_2$ in terms of minimizing $J$.

Although for signal $v_2$, I got a minimum $J = 5.7778$ when doing autoregressive spectrum estimation with order equals to 29, the autoregressive spectrum estimation is generally not suitable for signal $v_2$. The reason is that the result $J_{\text{min}} = 5.7778$ and order* = 29 can be obtained only when the true spectrum is provided. However, we will not have true spectrum when doing estimation. Therefore, we will not know which order is the best one. Same argument could be applied to signal $v_1$ using periodogram-based estimation with very large number of segments.

The power spectrum $P_1$ of signal $v_1$ is quiet smooth. That is, it can easily be
approximated by polynomial with low order. Therefore, the autoregressive spectrum estimation will yield a good result. Periodogram-based estimation provides higher resolution but higher variance. The high resolution is unnecessary and the high variance is undesired for P1. Thus, Periodogram-based estimation is less suitable for smooth power spectrum.

The power spectrum P2 of signal v2 is quiet sharp and has several peaks. We know that sharper function should be approximated by higher order polynomial. And from the discussion in part A.2, high order approximation is unreliable for autoregressive spectrum estimation. Therefore, we should avoid using autoregressive spectrum estimation for sharp and peaky spectrums. On the other hand, periodogram-based estimation can show us the clear peaks of the spectrum. In addition, the high resolution helps us locate the peaks more accurately.

If we aim only to identify and precisely locate sinusoidal components, we for sure should use periodogram-based estimation for sharp and peaky spectrums like P2. As for smooth spectrum like P1, since there is no peak in this kind of spectrum, I think the important thing is to know the allocation of energy among the frequency. In this aspect, autoregressive spectrum estimation gives clearer result for P1. Therefore, change of the criterion does not change the relative performance of the techniques.

A.4 Analysis of an Unknown Signal

First of all, we should try to use periodogram-based estimation to identify whether the spectrum is smooth or peaky. Here I use Bartlett’s method with 4 segments. The results are:

![Bartlett's method with 4 segments](image1)

![Bartlett's method with 4 segments](image2)

We find a huge peak in the periodogram of signal v3. This profile of spectrum could not be generated from autoregressive process with order eight. Therefore, we know that signal v3 is a realization of the WSS random process that comprises 3-5
sinusoidal components and white noise and signal v4 is a realization of the autoregressive process of order eight.

For signal v3, in order to identify sinusoidal component clearly, we desire higher resolution and lower variance. Therefore, I try to use Welch’s method to better the estimation. The result is shown below:

I am able to identify that the sinusoidal components are located about at frequency $0.6807\pi$ and $0.3213\pi$. Although there should be more components, the peak values are too small to be differentiated from noise.

For the signal v4, since we know that it is generated by an autoregressive process with order 8, it is intuitive to use autoregressive spectrum estimation with order 8 to estimate the spectrum. In addition, we find that the resulting estimation is quite smooth so we may also use Bartlett’s method with large S to generate good result. The resulting estimations of the two methods are shown below: