Part C: Designing a Rate-Conversion System

This part included the design of a rate-conversion system for a sampled analytical signal of frequency 48 kHz. The goal was to design a system that changes the sampling frequency from 48 kHz to 32 kHz, such that the mean-squared error (MSE) between the system output and the true signal $x_{32}$ as determined from the analytical continuous-time signal was minimized:

$$\text{MSE} = \frac{1}{N} \sum_{i=1}^{N} (y_i - x_{32}[i])^2$$

First of all, we realize that the sampling rate is reduced by a factor of 2/3, such that we design a system depicted as in Figure 11 with $L = 2$ and $M = 3$.

In order to design a filter and choose specifications for this filter such that the MSE is minimized, we considered several properties of the signal as well as the available filters before choosing the final specifications. This process is outlined below, after which the final specifications of our system design are given.

First of all, we investigated the properties of the underlying continuous-time signal. We realize that the signal consists of a superposition of two signals – a signal of frequency $f_1 = 5.1$ kHz and $f_2 = 9.8$ kHz. Hence, we concluded that the frequency spectrum of this signal consists roughly of two peaks at frequencies $f_1$ and $f_2$. In order not to get any aliasing but to catch the entire signal, we need to find an appropriate passband edge for the LP filter. Looking at the Fourier transform of the higher frequency envelope in MATLAB (using `fft`), we find that the signal has frequencies in the shape of a Gaussian curve. For inspection, we decided that all frequencies up to 10 kHz would need to be included. Thus, for safety, we chose a passband edge of 12 kHz (corresponding to $\pi/4$) for our filter.

For no aliasing, the cutoff frequency of low-pass filter is required to be $\pi/3$ for the specified up-and downsampling rates. For symmetry, we then chose a stopband edge of $5/12\pi$. In total, this gives a transition band of $\pi/6$.

For the filter choice, we considered the different properties of filter types designed in part B of this project.

Ideally, we desire a passband gain as close as possible to unity such that the signal is reproduced accurately. This would make both the Butterworth and the Chebyshev Type II filter good candidates as both have very little ripple in the passband. However, the Butterworth filter required a relatively high order to meet specifications in part B and here, we are restricted to use a maximum filter order of 20, such that we eliminated the Butterworth filter from our
Another desired property is a constant group delay for frequencies in the passband, i.e., \( \omega < \frac{n}{4} \), such that the final signal can be shifted to its original position during post-processing. If the group delay is not constant, it would be hard to apply any straightforward post-processing to shift samples to their original position. Comparing the group delays, we find that the Parks-McClellan filter is best for constant group delay. However, it also requires a very high filter order, which is a constraint in our design. Additionally, this filter has large ripples in the passband also, which might affect the signal more than any shift in the group delay compared to other filters. Both the remaining filters – Chebyshev Type II and Elliptical – have a relatively constant group delay for \( \omega < \frac{n}{4} \).

From these two filters, Chebyshev Type II has a flat passband, which makes it more desirable. However, the Elliptical filter allows for ripples in the pass- and stopband by taking variables for both these values in MATLAB. Additionally, the Elliptical filter is excellent at meeting specifications at low filter order.

Both these filters were explored more extensively in determining filter order but it was found that the Chebyshev Type II filter, with its passband closest to 0 dB, performed better than the Elliptical filter. Hence, our final design is based on choosing the Chebyshev Type II filter.

In choosing final specifications for the filter, the above cutoff frequencies and passband/stopband edge frequencies were taken as a basis and changed in several iterations. The minimum and maximum passband and stopband gain were estimated from initial values as given in part B but then altered in iterations of the filter design to minimize the MSE.

According to this method, the final specifications that were used in the filter design are as follows:

- Passband and stopband edge: \( \omega_p = \frac{1}{4} \pi \) and \( \omega_p = \frac{5}{12} \pi \)
- Width of transition band: \( \omega_t = \frac{1}{6} \pi \)
- Maximum gain in passband: 0 dB
- Minimum gain in passband: 1 dB
- Maximum gain in stopband: -31 dB

The optimal filter order for our specifications was then determined first by `cheb2ord` in order to get a rough estimate of a desirable filter order. This filter order was then increased to minimize the MSE, such that the final filter order used was \( n = 16 \).
In order to minimize the MSE between the signal output and the actual signal, we applied two post-processing steps on the signal. Firstly, in order to account for the group delay of the Chebyshev Type II filter for \( \omega < \frac{\pi}{4} \), we advanced the signal in time after filtering to compensate for the delay caused by the filter. The mean group delay for the frequencies in the passband was thus found but the final value of \( \tau_g = 3 \) was chosen, which was closer to the group delay of the higher frequency signal component. Hence, we zero-padded with 3 zeros before filtering and then discarded the first 3 samples of the signal after filtering in order to advance the signal in time by 3.

Additionally, it was necessary to account for amplitude scaling. The upsampling process introduces a scaling factor of \( 1/L = 1/2 \), which needs to be reversed in the final signal. Hence, we multiplied the output signal by a factor of 2 to compensate for this.

With the above specifications for the Chebyshev Type II filter design and post-processing steps, we calculated a mean-squared error of \( \text{MSE} = 7.9101 \times 10^{-4} \) between the output signal and the computed signal \( x_{32} \).