Problem 1. (40 Points)

Classify each one of the following as True/False. All “True” answers must be supported with a justification, and “False” answers with a counterexample. Each of the 10 questions is worth 4 points.

1. If two different basic feasible solutions of a linear program are optimal, then they must correspond to adjacent vertices of the feasible region.

2. The problem \( \max \sum_{j=1}^{n} c_j x_j \) subject to \( \sum_{j=1}^{n} a_j x_j \leq b \) with \( c_j, a_j \geq 0 \) can be modeled as a linear optimization problem.

3. Let \( P = \left\{ \sum_{i=1}^{k} \lambda_i x^i \mid \sum_{i=1}^{k} \lambda_i = 1, \lambda_i \geq 0, i = 1, \ldots, k \right\} \), where \( S = \{x^1, x^2, \ldots, x^k\} \) is a set of given vectors. Then the set of extreme points of \( P \) is \( S \).

4. If the dual of a linear optimization problem has a degenerate basic feasible solution, then the primal problem must have multiple optimal solutions.

5. The only way that the simplex method can get to a degenerate basic feasible solution at some iteration is when there is a tie in the min-ratio test in the previous iteration.

6. In a standard form linear optimization problem, degenerate basic feasible solutions result only when the linear optimization problem has redundant constraints.
7. If the dual has multiple optimal solutions, then all optimal primal basic feasible solutions are degenerate.

8. Every unbounded polyhedron must have at least one extreme ray.

9. If the projection of a set is convex, the set has to be convex.

10. If the reduced cost of a non-basic variable in an optimal basis is zero, then the BFS is degenerate.

Solution

1. FALSE. Let $c = 0$. Then every BFS is optimal, and in general every BFS is clearly not adjacent.

2. TRUE. Let $y_j = |x_j|, \forall j$ and a linear optimization problem in $y$ follows. We can then form an optimal solution by taking $x = y$, but any $x$ with $x_j = \pm y_j, \forall j$ will be a solution.

3. FALSE. Consider the case where $S$ consists of 3 distinct colinear points. Then the middle one is not an extreme point.

4. FALSE. This need only be true if the degenerate basic feasible solution of the dual is optimal and corresponds to two or more distinct bases, in which case the corresponding system of equations leads to multiple optima in the primal. As a counterexample, let the dual be unbounded with a degenerate BFS. Then the primal has no optimal solutions.

5. FALSE. Degeneracy can also occur when there is no tie but we move zero distance from an already degenerate solution.

6. FALSE. Consider a tetrahedron.

7. FALSE. We are only guaranteed of the existence of one such degenerate primal basic feasible solution. Not all of them need to be degenerate.

8. FALSE. Consider $\mathbb{R}^n$ which is a polyhedron without any extreme rays (refer to the definition of an extreme ray)

9. FALSE. A non-convex body in a higher dimension can have a convex projection.

10. FALSE. It is not the definition of degeneracy.
Problem 2. (30 Points)

(15 points) Consider the robust optimization problem

\[
\min_{\{x : Dx \geq f\}} \max_{\{c : Ac \leq b\}} c^T x,
\]

that is the feasible space is \(Dx \geq f\), but the objective function \(c\) is uncertain and the possible \(c\)'s satisfy: \(Ac \leq b\). Applying duality in the inner problem formulate the above problem as a linear optimization problem.

(15 points) Show that the following two statements are equivalent.

- Every vector such that \(Ax \geq 0\) and \(x \geq 0\) must satisfy \(x_1 = 0\).
- There exists some \(p\) such that \(p^T A \leq 0\), \(p \geq 0\), and \(p^T A_1 = -1\), where \(A_1\) is the first column of \(A\).

Solution:

(a)

Firstly, we assume the inner problem is feasible, otherwise the problem is not well-defined. Applying strong duality in the case where it has a finite optimal cost, the problem is

\[
\min_{\{x : Dx \geq f\}} \begin{cases} 
\min_{\{p : A^T p = x, p \geq 0\}} p^T b, & x \text{ is s.t. the inner problem is bounded,} \\
\infty, & x \text{ is s.t. the inner problem is unbounded}
\end{cases}
\]

So we have the following linear optimization problem

\[
\min_{p, x} \quad p^T b \\
\text{s.t.} \quad Dx \geq f, \\
A^T p - x = 0, \\
p \geq 0.
\]

(b)

This follows easily from the fact that a strictly complementary pair always exists. We can also prove this by considering the following primal dual pair

\[
\min -x_1 \\
\text{s.t.} \quad Ax \geq 0 \\
x \geq 0
\]

\[
\max 0p \\
\text{s.t.} \quad p^T A \leq -e_1 \\
p \geq 0
\]
Since $x_1 = 0$ always, the optimal solution is 0. Primal is feasible and optimal solution is 0, therefore dual has to be feasible. Hence, we have the existence of such a vector. We have shown one side. The other side is trivial.

Problem 3. (30 Points)

A company makes four products 1, 2, 3, and 4 and uses 3 resources A, B, and C. The company decides on the product mix by solving the following linear optimization problem:

$$
Z^* = \max \quad 16x_1 + 14x_2 + 15x_3 + 50x_4 \\
\text{s/t.} \quad \begin{align*}
2x_1 + 2x_2 + 5x_3 + 16x_4 & \leq 800 \quad (A) \\
3x_1 + 2x_2 + 2x_3 + 5x_4 & \leq 1000 \quad (B) \\
2x_1 + 1.2x_2 + x_3 + 4x_4 & \leq 680 \quad (C) \\
x_1, x_2, x_3, x_4 & \geq 0
\end{align*}
$$

The company solves the problem using the simplex method, and obtains the following optimal tableau:

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<td>1</td>
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</table>

(2 points) What is the optimal solution, and the optimal solution value?
(2 points) What is the optimal basis $B$, and what is $B^{-1}$?
(2 points) In one sentence, what is the optimal strategy?
(2 points) Is the optimal solution unique?
(4 points) What are the optimal dual variables?
(3 points) By how much should the profit of product 3 change, so that it is used in an optimal solution?
(5 points) What should the minimum profit of product 2 be, so that the company continues to produce it?
(3 points) Find the range on resource B, so that the current basis remains optimal.
(4 points) Suppose that resource B becomes $1000 + \theta$. Describe in as much detail as possible how the optimal profit changes with $\theta$.
(3 points) A new product requiring 4 units of resource A, 4 units of resource B and 1 unit of resource C is proposed. What should the profit of this product be in order to produce it?
Solution

(2 points) \( x' = (200, 200, 0, 0, 0, 0, 40) \), \( Z^* = 6000 \).

(2 points) 
\[
B = \begin{bmatrix}
2 & 2 & 0 \\
2 & 3 & 0 \\
1.2 & 2 & 1
\end{bmatrix}, \quad B^{-1} = \begin{bmatrix}
1.5 & -1 & 0 \\
-1 & 1 & 0 \\
0.2 & -0.8 & 1
\end{bmatrix}.
\]

(2 points) The company should produce 200 units of product 1, and 200 units of product 2, at a cost of \( 6000 \).

(2 points) Yes. Since the reduced cost of every nonbasic variable is negative (remember, we are maximizing), moving positive distance to “another optimal solution” must strictly decrease the cost, so this other optimal solution must not exist.

(4 points) 
\[
p' = c'B^{-1} = (14, 16, 0) \begin{bmatrix}
1.5 & -1 & 0 \\
-1 & 1 & 0 \\
0.2 & -0.8 & 1
\end{bmatrix} = (5, 2, 0)
\]

(3 points) 
\[
c_3 = c_3 - p'A_4 = c_3 - (5, 2, 0) \begin{bmatrix}5 \\ 2 \\ 1 \end{bmatrix} = c_3 - 29 \geq 0 \implies \Delta c_3 \geq 29 - 15 = 14.
\]

(5 points) Firstly,
\[
p' = c'B^{-1} = (c_2, 16, 0) \begin{bmatrix}
1.5 & -1 & 0 \\
-1 & 1 & 0 \\
0.2 & -0.8 & 1
\end{bmatrix} = (1.5c_2 - 16, 16 - c_2, 0)
\]
\[
c'_N = c'_N - p'N = (15, 50, 0, 0) - (1.5c_2 - 16, 16 - c_2, 0) \begin{bmatrix}5 \\ 2 \\ 1 \end{bmatrix} = (15, 50, 0, 0) - (7.5c_2 - 80 + 32 - 2c_2, 24c_2 - 256 + 80 - 5c_2, 1.5c_2 - 16, 16 - c_2)
\]
\[
= (15, 50, 0, 0) - (5.5c_2 - 48, 19c_2 - 176, 1.5c_2 - 16, 16 - c_2)
\]
\[
= (-5.5c_2 + 63, -19c_2 + 226, -1.5c_2 + 16, -16 + c_2) \leq 0 \implies c_2 \geq \frac{226}{19}.
\]

(3 points) We require
\[
B^{-1}b = \begin{bmatrix}
1.5 & -1 & 0 \\
-1 & 1 & 0 \\
0.2 & -0.8 & 1
\end{bmatrix} \begin{bmatrix}800 \\ B \\ 680 \end{bmatrix} = \begin{bmatrix}1200 - B \\ B - 800 \\ 160 - 0.8B + 680 \end{bmatrix} \geq 0
\]
\[
\implies 800 \leq B \leq 1050.
\]
(4 points) The optimal profit in a linear maximization problem is a concave function in the right-hand-side. For $800 \leq 1000 + \theta \leq 1050$, i.e. $-200 \leq \theta \leq 50$, our basis remains optimal, and hence so do our dual variables, so this function is

$$f(b) = p^*b = (5, 2, 0) \begin{bmatrix} 800 \\ 1000 + \theta \\ 680 \end{bmatrix} = 4000 + 2(1000 + \theta) = 6000 + 2\theta.$$ 

Outside of this range the optimal basis will change and we cannot give any more detail without solving for $p^*$ for some such $b$.

(3 points) These resources are valued at

$$p^*b = (5, 2, 0) \begin{bmatrix} 4 \\ 4 \\ 1 \end{bmatrix} = 28,$$

so the profit of this product must be at least 28 in order to produce it.