Definition 1. A \((l, \epsilon)\)-unbalanced expander is a bipartite simple graph \(G = (U, V, E)\), \(|U| = n, |V| = m\), with left degree \(d\) such that for any \(X \subset U\) with \(|X| \leq l\), the set of neighbors \(N(X)\) of \(X\) has size \(|N(X)| \geq (1 - \epsilon)d|X|\).

We also define \(E(X : Y) = E \cap (X \times Y)\) to be the set of edges between the sets \(X\) and \(Y\).

The following well-known proposition can be shown using Chernoff bounds.

Claim 1. For any \(n/2 \geq l \geq 1, \epsilon > 0\), there exists a \((l, \epsilon)\)-unbalanced expander with left degree \(d = O(\log(n/l)/\epsilon)\) and right set size \(O(ld/\epsilon) = O(l \log(n/l)/\epsilon^2)\).

Now we show that the expander matrices have the null-space property. Let \(A\) be an \(m \times n\) adjacency matrix of an unbalanced \((2k, \epsilon)\)-expander \(G\) with left degree \(d\). Let \(\alpha(\epsilon) = (2\epsilon)/(1 - 2\epsilon)\).

Lemma 1. Consider any \(\eta \in \mathbb{R}^n\) such that \(A\eta = 0\), and let \(S\) be any set of \(k\) coordinates of \(\eta\). Then we have

\[
\|\eta_S\|_1 \leq \alpha(\epsilon)\|\eta\|_1
\]

Proof. Without loss of generality, we can assume that \(S\) consists of the largest (in magnitude) coefficients of \(\eta\). We partition coordinates into sets \(S_0, S_1, S_2, \ldots, S_l\), such that (i) the coordinates in the set \(S_i\) are not-larger (in magnitude) than the coordinates in the set \(S_{i-1}\), \(i \geq 1\), and (ii) all sets but \(S_l\) have size \(k\). Therefore, \(S_0 = S\). Let \(A'\) be a submatrix of \(A\) containing rows from \(N(S)\).

The basic idea of the proof is as follows. Assume (by contradiction) that \(\|\eta_S\|_1\) is "large" compared to \(\|\eta\|_1\), which (by RIP1) implies that \(\|A'\eta_S\|_1\) is "large". Since \(0 = \|A'\eta\|_1 = \|A'\eta_S + A'\eta_{-S}\|_1\), it follows that \(\|A'\eta_{-S}\|_1\) must be "large", to cancel the contribution of \(A\eta_S\). The only way for this to happen though is if there are many edges in \(G\) from \(-S\) to \(N(S)\). This however would mean that the neighborhoods of \(S\) and blocks \(S_i\) have large overlaps, which cannot happen since the graph is an expander.

The formal proof follows.

From the RIP-1 property we know that \(\|A'\eta_S\|_1 = \|A\eta_S\|_1 \geq d(1 - 2\epsilon)\|\eta_S\|_1\). At the same time, we know that \(\|A'\eta\|_1 = 0\). Therefore

\[
0 = \|A'\eta\|_1 \geq \|A'\eta_S\|_1 - \sum_{l \geq 1} \sum_{(i,j) \in E, i \in S_l, j \in N(S)} |\eta_i| \\
\geq d(1 - 2\epsilon)\|\eta_S\|_1 - \sum_{l \geq 1} |E(S_l : N(S))| \min_{i \in S_{l-1}} |\eta_i| \\
\geq d(1 - 2\epsilon)\|\eta_S\|_1 - \sum_{l \geq 1} |E(S_l : N(S))| \cdot \|\eta_{S_{l-1}}\|_1/k
\]

From the expansion properties of \(G\) it follows that, for \(l \geq 1\), we have \(|N(S \cup S_l)| \geq d(1 - \epsilon)|S \cup S_l|\). It follows that at most \(dek\) edges can cross from \(S_l\) to \(N(S)\), and therefore

\[
0 \geq d(1 - 2\epsilon)\|\eta_S\|_1 - \sum_{l \geq 1} |E(S_l : N(S))| \cdot \|\eta_{S_{l-1}}\|_1/k \\
\geq d(1 - 2\epsilon)\|\eta_S\|_1 - d\epsilon k \sum_{l \geq 1} \|\eta_{S_{l-1}}\|_1/k \\
\geq d(1 - 2\epsilon)\|\eta_S\|_1 - d\epsilon \|\eta\|_1
\]

It follows that \(d(1 - 2\epsilon)\|\eta_S\|_1 \leq 2d\epsilon \|\eta\|_1\), and thus \(\|\eta_S\|_1 \leq (2\epsilon)/(1 - 2\epsilon)\|\eta\|_1\). \(\square\)