Lecture 17

NP Completeness + Reductions

- Overview of Cook's Thm
- $3\text{SAT} \equiv \text{Clique}$
Overview of Cook's Thm

$x_1 \ L \ R \ \rightarrow \ \text{CSAT}$

Since $L \in \mathbf{NP}$, there is $V$ which checks certificates for $A$'s "yes" instances.

Configuration of $V$ at step $i$:

$C_i$: Program Counter | auxiliary state | working storage

How do we describe?

Use lots of variables.

e.g. $Q^a_i$ "at time $i$, $PC = q$"

only constant number of options

only poly many options

$S^a_{ij}$ "at time $i$, $j$th memory location is a"

+ others to describe auxiliary state
The clauses:

1. Use Boolean logic to check validity at each step
   - exactly one $Q^i_{xj}$ is "on" for each $i,j$
   - initial configuration corresponds to start instruction, input $X$ in memory
   - final configuration corresponds to "PASS"

2. Use Boolean logic to check consistency between steps
   - compare $Q^i_{xj}$ vs $Q^i_{xj'}$ - is it possible transition for $V$?
     - e.g. $V$s Program never jumps from instruction 10 to instruction 2,3
       - configurations shouldn't either!

\[ S_{xj} = 1 \]

\[ \text{C}_0 \quad \text{PC} \quad \text{aux state} \quad \text{storage} \]

\[ V \]

\[ \text{C}_1 \quad \text{PC} \quad \text{aux state} \quad \text{storage} \]

\[ V \]

\[ \text{C}_2 \quad \text{PC} \quad \text{aux state} \quad \text{storage} \]

\[ \vdots \]

\[ \text{C}_{T(n)} \quad \text{PC} \quad \text{aux state} \quad \text{storage} \]
3-CNF SAT (often called just "3 SAT")

Terms:
- literal - occurrence of Boolean var or negation
  i.e. \( x_i \) or \( \overline{x}_i \)

Clause - "OR" of literals
i.e. \( x_1 \lor x_2 \lor \overline{x}_3 \)

CNF - conjunctive normal form
"AND" of clauses
i.e. \( (x_1 \lor x_2 \lor \overline{x}_3)(\overline{x}_1 \lor x_2)(x_3 \lor x_4) \)

3-CNF - each clause has exactly 3 distinct literals
i.e. \( (x_1 \lor x_2 \lor \overline{x}_3)(\overline{x}_1 \lor x_2 \lor x_3) \)

3 SAT:

Input: 3-CNF formula \( \Phi \) with \( n \) vars, \( m \) clauses.

Output: is \( \Phi \) satisfiable?

Thm: 3-SAT is NP-complete
Clique:

Input: $G = (V, E)$ graph, undirected
      $K$ integer

Output: Is there a clique of size $\geq K$?

$\begin{align*}
\text{subset } C \subseteq V \\
\text{s.t. } \forall u, v \in C \\
\quad (u, v) \in E
\end{align*}$

e.g.

Thm: Clique is NP-complete

Pf:

1) Clique is in NP
   Certificate = $C$

   Verifier checks:
   \begin{align*}
   |C| \geq K \\
   \forall u, v \in C \\
   (u, v) \in E
   \end{align*}

   $O(N^2)$ time
2) \(3SAT \leq CLIQUE\)  

Given \(3SAT\) formula \(\varphi = C_1 \lor C_2 \lor \ldots \lor C_k\) 
\(C_i = l_i^r \lor l_i^s \lor l_i^s\) for \(l_i \in \{x_1, \ldots, x_n\}\), 

Need to construct \(G_\varphi, K_\varphi\) s.t. \(\varphi\) is sat iff \(G_\varphi\) has Clique of size \(K_\varphi\).

Reduction:

Construct \(G_\varphi\): For each \(C_r = l_i^r \lor l_i^s \lor l_i^s\), insert new vertices \(V_i^r, V_i^s, V_i^s\) into \(V_\varphi\), put edge between \(V_i^r + V_j^s\) if
- \(r \neq s\)
- literals corresponding to \(V_i^r + V_j^s\), i.e., \(l_i^r + l_j^s\), are not negations (i.e., \(X_b \neq \overline{X_b}\))

Construct \(K_\varphi\): \(K_\varphi = K\)
Example

\[ \phi = (x_1 \lor x_2 \lor x_3) (\overline{x}_1 \lor x_2 \lor x_3) (x_1 \lor \overline{x}_2 \lor x_3) \]

Claim \( \phi \) satisfiable iff \( G_\phi \) has \( K_3 \)-clique

pf.

\( (\Rightarrow) \) take assignment satisfying \( \phi \)

\[ \begin{align*}
\text{e.g. } & x_1 = T, \ x_2 = T, \ x_3 = F \\
\end{align*} \]

\( C \leftarrow \) vertex corresponding to any one satisfied literal per clause.

\( C \) is a clique of size \( k = k_\phi \) : only missing edge no assignment sets both \( x_i \) or \( \overline{x}_i \) to true.
(\iffalse)

\begin{align*}
\text{take clique of size } K &= K_0 \\
\text{set each corresponding literal to "true"}
\end{align*}

\begin{align*}
\text{e.g. } (c_1, x_2), (c_2, x_2), (c_3, x_3) \\
X_2 &\leftarrow T, \bar{X}_3 \leftarrow T \\
\downarrow \\
X_2 &\leftarrow F
\end{align*}

\text{Setting is consistent}\hspace{1cm} \text{(since no edge causes } X_i = T \Rightarrow X_i = F)\hspace{1cm} \text{and satisfies all } K = K_0 \text{ clauses}

\begin{align*}
\text{Claim: Reduction is poly-time}
\end{align*}

\begin{align*}
\text{Clique is } NP\text{-hard}
\end{align*}

\begin{align*}
\text{Clique is } NP\text{-complete}
\end{align*}