A Converse for the Wideband Relay Channel with Physically Degraded Broadcastl

Nadia Fawaz & Muriel Médard
{nfawaz,medard}@mit.edu

MIT/RLE
Network coding and Reliable Communications Group

ITW Paraty
Low SNR/Wideband Regime

Wideband/Low SNR regime

- Available power shared among a large number of degrees of freedom
  \[ \Rightarrow \text{low SNR per degree of freedom} \]
- Performance power-limited, not interference-limited

Point-to-point Channel

- **AWGN channel capacity** \(\text{Shannon-1949:}\)
  \[ C_{AWGN} = \frac{P}{N_0} = \lim_{W \to \infty} W \log(1 + \frac{P}{WN_0}) \]
- **Non-coherent multipath fading channel capacity:**
  \[ C_{\text{Fading}} = \frac{P}{N_0} = C_{AWGN} \]
  Achieved with low duty-cycle peaky signaling

- **SIMO/MISO channel** \(\text{Zheng-Tse-2002:}\)
  \[ C_{SIMO} = (1 + |a|^2) \frac{P}{N_0} \quad / \quad C_{MISO} = \max(1, |b|^2) \frac{P}{N_0} \]

Figure:
SIMO/MISO channel
Wideband Relay Channel - Cut-set upper-bound and DF lower-bound

FD-AWGN relay channel capacity bounds Cover-ElGamal-1979, ElGamal-Mohseni-ITtrans2006

Wideband limit

- Cut-set upper-bound: \( C \leq \min \left\{ (1 + a^2) \frac{P_s}{N_0}, (1 + b^2 \gamma) \frac{P_s}{N_0} \right\} \)

- Generalized block-Markov lower bound: \( C \geq \min \left\{ \max\{1, a^2\} \frac{P_s}{N_0}, (1 + b^2 \gamma) \frac{P_s}{N_0} \right\} \)

FD non-coherent multipath fading relay channel Fawaz-Médard-ISIT2010

- Achievable hypergraph model

- Hypergraph min-cut: achieved by peaky binning DF relaying scheme

\[
R = \min \left\{ \max\{1, a^2\} \frac{P_s}{N_0}, (1 + b^2 \gamma) \frac{P_s}{N_0} \right\} \tag{1}
\]
Achievable Rates and Upper-bound

![Achievable Rates and Upper-bound](image_url)

**Figure:** Communication rate in the wideband AWGN/non-coherent multipath fading relay channel \((1 \leq \gamma b^2)\)
Problem formulation

• **Open question**: Can the gap to the cut-set upper-bound be closed?

• **Observations**:

  ![SIMO Channel Diagram]

  **Figure**: SIMO channel

- An $\infty$ capacity on the link R-D would be sufficient to achieve the cut $\frac{P}{N_0}(1 + a^2)$ like in SIMO.
- Because of power limit at relay, it cannot make its observation fully available to destination.

- **Converse**: when the broadcast channel component (BC) of the FD-AWGN relay channel is physically degraded, or stochastically degraded but treated as physically degraded by the source, then
  - the cut-set upper-bound cannot be reached in the low-SNR regime
  - selective DF optimal in the low-SNR
  - capacity is given by a simple hypergraph model
Converse proof outline

- Split total mutual information in two parts:
  - contribution from relay
  - remaining contribution from source after deducting contribution from relay
- Bound contributions using rate distortion
- Analyze interaction between 2 contributions using properties of rate distortion function

Figure: FD-AWGN relay channel
Converse: sketch of proof - contributions of different parts of network

- Total mutual information

\[ I(W; Y) \leq I(X; Y) = I(X; Y_2) + I(X; Y_3|Y_2) \]

- Relay contribution

\[ I(X; Y_2) \leq I(X; X_1) \leq I(X; \hat{X}) \leq I(X; Y_1) \leq n \log \left( 1 + a^2 \frac{P_s}{W_S N_0} \right). \]

\[ I(X; Y_2) \leq I(X_1; Y_2) \leq n \log \left( 1 + b^2 \gamma \frac{P_s}{W_S N_0} \right). \]

\[ I(X; \hat{X}) = h(X) - h(\tilde{X} | \hat{X}) \Rightarrow D \]

- Direct source contribution

\[ I(X; Y_3|Y_2) \leq I(X; Y_3) \leq n \log \left( 1 + \frac{P_s}{W_S N_0} \right). \]

\[ I(X; Y_3|Y_2) = h(\hat{X} + \tilde{X} + Z_3|X_1 + Z_2) - h(Z_3) \Rightarrow D, D_1 \]
Converse: sketch of proof - case where the relay cannot decode, information at relay

\[
\begin{align*}
  \mathbf{y}_1 &= a\mathbf{x} + \mathbf{z}_1 \rightarrow \hat{\mathbf{x}} \rightarrow \mathbf{x}_1 \\
  \gamma P \\
  \mathbf{y}_2 &= b\mathbf{x}_1 + \mathbf{z}_2 \\
  \mathbf{y}_3 &= \mathbf{x} + \mathbf{z}_3
\end{align*}
\]

Figure: FD-AWGN relay channel

- Let us consider the case where S tx at \( R > C_{SR} = W_S \log \left(1 + a^2 \frac{P_S}{W_S N_0}\right) \) \( \Rightarrow \) Relay cannot decode.
- \( \forall \) estimator \( \hat{\mathbf{x}} \): \( D \geq D_{\text{min}} = D_{\text{MMSE}} \)
- MMSE estimator: \( \hat{\mathbf{x}}_{\text{MMSE}}(\mathbf{y}_1) = E[\mathbf{x}|\mathbf{y}_1] \triangleq \int \mathbf{x} p(\mathbf{x}|\mathbf{y}_1) d\mathbf{x} \) orthogonality property: error \( \tilde{\mathbf{x}} \) orthogonal to observation \( \mathbf{y}_1 \)

\[
E[(\hat{\mathbf{x}} - \mathbf{x})g(\mathbf{y}_1)] = 0, \text{ for all functions } g.
\] (2)

- MMSE unbiased:
  \[
  E[\hat{\mathbf{x}}] = 0 \Rightarrow E[\hat{\mathbf{x}}_{\text{MMSE}}^H \hat{\mathbf{x}}_{\text{MMSE}}] = 0 = E[\hat{\mathbf{x}}_{\text{MMSE}}^H] E[\hat{\mathbf{x}}_{\text{MMSE}}] \text{ uncorrelated error and estimate} \]
Converse: sketch of proof - from source to relay

• Use i.i.d. Gaussian input $X_G \sim \mathcal{CN}(0, P_S I_n)$. justification: equivalence theory (Kötter-Effros-Médard-ITW2009, ITtrans2011)

• MMSE estimator: linear function of observation

\[
\hat{X}_{\text{MMSE}}(Y_1) = \frac{aP_S}{a^2P_S + W_SN_0} Y_1 \quad C_{\hat{X}\hat{X}} = \frac{a^2P_S}{a^2P_S + W_SN_0} P_S I_n.
\]

\[
\tilde{X}_{\text{MMSE}} \sim \mathcal{CN}(\mu_{\tilde{X}}, C_{\tilde{X}\tilde{X}}) \quad C_{\tilde{X}\tilde{X}} = \frac{P_S W_SN_0}{a^2P_S + W_SN_0} I_n
\]

\[
\hat{X} \perp \tilde{X}, \quad D_{\text{MMSE}} = \text{tr}(C_{\tilde{X}\tilde{X}}) = n \frac{P_S W_SN_0}{a^2P_S + W_SN_0}
\]

\[
l(X_G; \hat{X}_{\text{MMSE}}) = l(X_G; Y_1)
\]

• $\infty$-bandwidth $W_S$ limit:

\[
D_{\text{min}} \sim_{W_S \to +\infty} nP_S.
\]

• Interpretation: for an other estimator: $D \geq D_{\text{min}} \rightarrow$ maximum distortion. In the limit of a large bandwidth, if the relay cannot decode, large noise power $W_SN_0$ impairs estimation.
Converse: sketch of proof - from relay to destination

![Diagram](image)

• Upper bound $r - d$ link by an error-free bit-pipe of capacity
  
  $C_{rd} + \varepsilon_2 = W_r \log \left(1 + \frac{b^2 P_r}{W_r N_0}\right) + \varepsilon_2$ from equivalence theory

  (Kötter-Effros-Médard-ITW2009, ITtrans2011)

• We need to send $\hat{X} = \hat{U} + \tilde{U}$ over error-free tube with finite capacity

  $$\frac{W_s R_{\hat{U}}(D_{\hat{U}})}{n} \leq C_{rd} + \varepsilon_2$$

  $$\frac{D_{\hat{U}}}{n} \geq \frac{a^2 P_s}{a^2 P_s + W_s N_0} P_s \exp \left(-\frac{W_r}{W_s} \log_e \left(1 + \frac{b^2 P_r}{W_r N_0}\right)\right)$$

  
  
  $$\frac{W_s}{n} I(X_G; \hat{U}) \leq \frac{a^2 b^2 P_s P_r}{W_s N_0^2} \rightarrow W_s, W_r \rightarrow +\infty \ 0$$

• Interpretation: In the limit of a large bandwidth, if the relay cannot decode, large noise power $W_s N_0$ and finite $r - d$ link capacity render the relay contribution useless.
Converse: sketch of proof - remaining contribution from source

• **Bounds on source contribution**: if relay cannot decode,

\[
W_s \log \left( 1 + \frac{P_S}{W_s N_0 \left( 1 + \frac{a^2 P_S}{W_s N_0} \right)} \right) \leq \frac{W_s}{n} I(X; Y_3|Y_2) \leq W_s \log \left( 1 + \frac{P_S}{W_s N_0} \right)
\]

\[
\lim_{W_s \to +\infty} \frac{W_s}{n} I(X; Y_3|Y_2) = \frac{P_S}{N_0}, \quad \forall n \in \mathbb{N}^*
\]

• We now use the fact that the BC channel is physically degraded or treated as such by the source

• We may then assume that the source input is a Gaussian - allows to achieve its Shannon-capacity with an arbitrarily small error probability.

• The s-r link with a Gaussian input is thus equivalent to an error-free link

• The source input and the relay observation are jointly Gaussian

• **Interpretation**: In the limit of a large bandwidth, if the relay cannot decode, the source contribution can saturate the direct link $s - d$. 
Conclusion and Perspectives

- **Conclusions:** In the wideband/low-SNR regime,
  - Neither interference nor fading are issues
  - Relay should not AF but DF
  - Wireless relay channel has equivalent wired network hypergraph model
  - Network coding should occur in the digital domain over the PHY relaying hypergraph
  - Hypergraph min-cut achieved by non-coherent peaky binning PHY relaying combined with digital network coding

- **Future work**
  - Extension to larger networks
  - Relay positioning in the low SNR regime

Thakur-Fawaz-Médard-INFOCOM2011, ISIT2011