1. Spherical Waves in the Sinusoidal Steady State

A. Spherical Coordinates & Symmetry

In the sinusoidal steady state:

\[ p(r, t) = \text{Re}\left\{ P(r) e^{j\omega t} \right\} , \quad (4.1) \]

where for forward going waves:

\[ P(r) = \frac{A}{r} e^{-jkr}. \quad (4.2) \]

Comparing this description of the pressure term to the description of the outward going wave in a plane wave, \( P(x) = P^+ e^{-jkx} \), note that while the complex amplitude \( P^+ \) has units of pressure, \( A \) has units of pressure times length.

Based on the conservation of mass, the phaser describing the spatial dependence of the velocity is

\[ V(r) = -\frac{1}{j\omega \rho_0} \frac{\partial P(r)}{\partial r} , \]

where acceleration of particles in the positive direction produces a reduction in local sound pressure.

The partial derivative of \( P(r) \) with respect to \( r \) can be computed in terms of the chain rule:

\[
\begin{align*}
\frac{\partial P(r)}{\partial r} &= \frac{d(A/r)}{dr} e^{-jkr} + \frac{A}{r} \frac{d(e^{-jkr})}{dr} \\
&= -\frac{A}{r^2} e^{-jkr} - jk \frac{A}{r} e^{-jkr} \\
&= -\frac{A}{r} e^{-jkr} \left( \frac{1}{r} + jk \right) \\
&= -P(r) \left( \frac{1}{r} + jk \right),
\end{align*}
\]

and the phaser describing velocity equals:
\[ V(r) = \frac{P(r)}{j\omega \rho_0} \left( \frac{1}{r} + jk \right), \quad \text{and} \]
\[ V(r) = \frac{P(r)}{j\omega \rho_0} \left( \frac{1}{r} + jk \right) = \frac{P(r)}{\omega \rho_0} \left( \frac{k}{jkr} + 1 \right) = \frac{P(r)}{\rho_0 c} \left( \frac{1}{jkr} + 1 \right). \quad (4.3) \]

The particle velocity, for a given source amplitude, varies with distance from the source \( r \), in a nonlinear manner and also varies with wave number \( k \) (unlike uniform plane waves).

The Specific Acoustic impedance in the spherical wave is then:
\[ Z_s^S(r) = \frac{P(r)}{V(r)} = \frac{P(r)}{\frac{P(r)}{\rho_0 c} \left( \frac{1}{jkr} + 1 \right)} = \frac{\rho_0 c}{\frac{1}{jkr} + 1}. \quad (4.4) \]

The specific acoustic impedance (magnitude and angle) seen by the wave depends on \( r \) and \( k \):
\[ Z_s^S(r) = \frac{P(r)}{V(r)} = \frac{\frac{A}{r} e^{-jkr}}{\frac{1}{z_0} e^{-jkr} \left( 1 + \frac{1}{jkr} \right)} = \frac{z_0}{1 + \frac{1}{jkr}}. \quad (4.5) \]
Specific Acoustic Impedance vs r: 
frequency=200 Hz

\[ Z_s^S(r) = \frac{P(r)}{V(r)} = \frac{z_0}{1 + \frac{1}{jkr}} \]

In the “Far Field”, where \( kr >> 1 \), Equations 4.4 and 4.5 are greatly simplified:

\[ Z_s^S(r) \bigg|_{kr >> 1} = \rho_0 c \quad \text{and} \quad V(r) \bigg|_{kr >> 1} = \frac{A}{r} e^{-jkr} \left( \frac{1}{\rho_0 c} \right) \frac{P(r)}{\rho_0 c}. \tag{4.6} \]

In the “Far Field” \( V(r) \) and \( P(r) \) are proportionately related by the characteristic impedance of the medium \( z_0 = \rho_0 c \text{ as in a uniform plane wave} \), and the magnitudes of \( V(r) \) and \( P(r) \) decrease proportionately with distance from the source.

The inverse proportionality between \( r \) and \( |V(r)| \) and \( |P(r)| \) in the far field leads to an average power density (or sound intensity) that decreases as the square of \( r \):

\[ \bar{I}(r) \bigg|_{kr >> 1} = \frac{1}{2} \text{Re} \left\{ \frac{P(r)V^*(r)}{z_0} \right\} = \frac{1}{2} \left| \frac{P(r)}{z_0} \right|^2 = \frac{1}{2} \left| V(r) \right|^2 z_0 = \frac{1}{2} \frac{|A|^2}{z_0 r^2}. \tag{4.7} \]

This relationship is often referred to as the “inverse square law”.

In the “Near Field” where \( kr << 1 \), \( Z_s^S(r) \) is approximately masslike:

\[ Z_s^S(r) = \frac{z_0}{1 + \frac{1}{jkr}} = \frac{jz_0 kr}{jkr + 1} \]

\[ Z_s^S(r) \bigg|_{kr << 1} = jz_0 kr. \tag{4.8} \]
and the particle velocity lags the sound pressure by $\pi/2$ radians:

$$V(r)|_{kr<1} = \frac{P(r)}{Z^S} = \frac{P(r)}{j\omega kr}. \quad (4.9)$$

Since $Z^S(r)$ is dominated by a reactive term when $kr<1$, little power is transferred from the source to the space that surrounds it.

2. "Simple" Spherical Sources:

A. Pulsing Sphere

Fig 3.2 A pulsing sphere

Where simple means all parts of the surface are vibrating in phase! The sphere pulsations are also constrained to be small compared to the steady-state dimensions.

"Source Strength" = $U_S = 4\pi a^2 V(a)$. \quad (4.10)

(Note that Source Strength is a ‘volume velocity’ with units of m$^3$/s that results from the product of the uniform particle velocity of the source surface and the surface area.)

The “Radiation impedance” is the Acoustic Impedance (with units of Acoustic Ohms Pa-s/m$^3$) at the surface of the source is:

$$Z(a) = \frac{P(a)}{U_S} = \frac{1}{4\pi a^2} \frac{z_0}{1 + \frac{1}{jka}} = \frac{z_0}{4\pi a^2} \frac{jka}{1 + jka}. \quad (4.11)$$

In the High Frequencies, $ka \gg 1$, $Z(a)$ looks like a characteristic acoustic impedance:

$$Z(a) \approx \frac{z_0}{4\pi a^2}. \quad (4.12)$$

At Low Frequencies, when $ka \ll 1$, and

$$Z_R = \frac{P(a)}{U_S} = \frac{z_0}{4\pi a^2} \frac{jka}{1 + jka} \left(1 - jka\right) = \frac{z_0}{4\pi a^2} \frac{(ka)^2 + jka}{\left(ka\right)^2 + 1} = \frac{jka z_0}{4\pi a^2}. \quad (4.13)$$

Since the real part of $Z(a)$ in the “Low Frequencies” is proportional to $\omega^2$, the average power radiated for a given source strength is proportional to frequency squared.

$$\Pi = \frac{1}{2} |U_S|^2 \text{Re} \{Z(a)\} = |U_S|^2 \frac{z_0}{8\pi a^2} \frac{(ka)^2}{\omega^2} = |U_S|^2 \frac{\omega^2 z_0}{8\pi c^2}. \quad (4.14)$$

Therefore, at low frequency relatively little average sound power radiates to the environment. Also note that for a given $U_S$ with $ka \ll 1$ the average power radiated is independent of the dimensions of the source!
B. Description of wave propagation in terms of source strength

We have described the sound pressure in a spherical wave in terms of a complex constant $A$, i.e.

$$P(r) = \frac{A}{r} e^{-jkr}.$$  

Knowing the source strength, we can define $A$ in terms of the sound pressure at the walls of the spherical source (remember the source has a radius of $a$):

$$P(a) = \frac{A}{a} e^{-jka} = U_S Z(a); \text{ i.e. } A = a U_S Z(a) e^{jka}. \quad (4.15)$$

In the Low Frequency situation, i.e. $ka << 1$, we can approximate $e^{jka}$ as 1, and we can use the Low Frequency approximation for $Z(a)$:

$$A = a U_S z_0 4 \frac{ka}{4\pi a^2} (ka)^2 + jka$$

Therefore, when the radius $a$ of the source is such that that $ka << 1$, the sound pressure at some distance $r$ is:

$$P(r) = j \omega U_S \frac{\rho_0}{4\pi r} e^{-jkr}. \quad (4.17)$$

3. More About Radiation Impedance

We have just argued that the specific acoustic impedance which describes the relationship between sound pressure and particle velocity is the same in the far field for any 'simple' source. However, one constraint on sound radiation that differs for the four simple sources in Figures 3.3 and 3.4 is the load that the surrounding air places on the radiators, i.e. the radiation impedance $Z_R$. Knowledge of $Z_R$ allows us to quantify:

1. Power radiated from a source to the environment, and
2. The resistive and reactive forces of the medium on the source.

The pulsing sphere revisited:

We have already derived the radiation impedance acting on the surface of a pulsing sphere of radius $a$, where we can modify (3.23) such that:

$$Z_R = \frac{P(a)}{U_S} = \frac{z_0}{4\pi a^2} \frac{ka}{1 + jka} \left(1 - jka \right) = \frac{z_0}{4\pi a^2} \frac{(ka)^2 + jka}{(ka)^2 + 1}.$$  

Eqn. 4.18 describes a real part and an imaginary part to $Z_R$ where $Z_R = R_R + jX_R$, and

$$R_R = \frac{z_0 (ka)^2}{4\pi a^2 \left((ka)^2 + 1\right)}, \text{ and } X_R = \frac{z_0 ka}{4\pi a^2 \left((ka)^2 + 1\right)}. \quad (4.19)$$

According to (3.33), at low frequencies when $ka << 1$, the radiation resistance is independent of the sphere's radius and has a magnitude that increases as the square of $\omega$:

$$R_R|_{ka<<1} = \frac{z_0 (ka)^2}{4\pi a^2} = \frac{z_0 k^2}{4\pi} = \frac{z_0 \omega^2}{4\pi c^2}. \quad (4.20)$$

In the same low-frequency range, the radiation reactance is positive and proportional to frequency and is well approximated by an acoustic mass or inertance:
This mass is equivalent to a blanket of air around the sphere of thickness $a$.

At high frequencies, $ka \gg 1$, the radiation resistance approximates the ratio of the characteristic impedance of the medium and the area of the sphere and the reactance decreases proportionately with sound frequency:

$$R_R\big|_{ka\gg1} = \frac{z_0}{4\pi a^2}, \text{ and } X_R\big|_{ka\gg1} = \frac{z_0}{4\pi a^2 ka} = \frac{\rho_0 c^2}{4\pi a^3 \omega}.$$  \hspace{1cm} (4.22)

We can see this low-frequency mass dominance and high-frequency resistance dominance in Figure 4.5.

Figure 4.5: The normalized radiation resistance $R_N$ and reactance $X_N$ acting on a pulsating sphere. The normalization factor depends on the surface area of the sphere $S$ and the characteristic impedance of the media $z_0$. The dashed lines illustrate the slopes of relationships that are proportional to $\omega$, $\omega^2$ and $1/\omega$.

Each of the impedance components described above have a non-simple frequency dependence. There is trick to thinking about these in a more simple way. The radiation admittance of a sphere is much simpler in form, where

$$Y_R = \frac{1}{Z_R} = \frac{1}{R_{Y_R}} + \frac{1}{jX_{Y_R}},$$  \hspace{1cm} (4.23)

where: $R_{Y_R} = \frac{z_0}{4\pi a^2}$, and $X_{Y_R} = \omega \frac{\rho_0 a}{4\pi a^2}$.

When $X_{Y_R} < R_{Y_R}$ the radiation admittance is dominated by the reactance and vice versa. Eqn 4.23 is equivalent to the two impedance components being placed in parallel with each other. Such parallel circuits will be better discussed later in the course.

More discussions of the radiation impedance can be found in Beranek 1986.
4. Generalization of the simple source concept

The sound radiated from an acoustically small source with \( kx \ll 1 \), where \( x \) is some descriptive linear dimension of the source, can be characterized by a source strength \( U_S \) as long as all parts of the 'radiator' move in phase.

For example the output of three small loud speakers - of diaphragm areas \( S_1, S_2 \) and \( S_3 \) and diaphragm velocities \( V_1, V_2, V_3 \), that all fit within an imaginary sphere of radius \( a \) can be approximated by the output of a simple source with source strength:

\[
U_S = \sum_{i=1}^{n} S_i V_i = \int_S \overline{v}(S) \cdot dS
\]

As long as all parts of all of the diaphragms are moving in phase and \( ka \ll 1 \).

Other low-frequency "simple sources" include:

A Loud Speaker in a box,

The open end of an organ pipe,

Radiation from the mouth.

The equivalence to a simple source when \( ka \ll 1 \) also implies that far away from the radiator in the Far Field, where \( r \gg a \), the radiation is spherically symmetric and the sound pressures and particle velocities within the wave are quantifiable in terms of the source strength and the Far Field produced by a spherical source:

\[
\begin{align*}
P(r) &= \frac{A}{r} e^{-jkr}, \\
\overline{v}(r) &= \frac{A}{\rho_0 c r} e^{-jkr}, \\
A &= \frac{j \omega U_S \rho_0}{4 \pi} \\
&\text{and } z(r) \approx z_0.
\end{align*}
\]  

\[(4.24)\]
5. Room Acoustics (Chapter 13 of Kinsler et al.)

a. The influence of reverberations on sound pressure and volume velocity

The importance of reflections in an acoustic environment is primarily dependent on the sound absorptive properties of the environment’s boundaries and furnishings. Anechoic environments are bounded by surfaces that are matched to the characteristic impedance of air. Such surfaces absorb all incident sound waves, such that there are no reflections. Rigid boundaries absorb little sound energy and cause large reflections. If the walls and all of the furnishings within a room were completely rigid and air were a truly lossless medium, sound emitted within such an reverberant environment would theoretically continue forever, bouncing from surface to surface without loss.

b. The Sabine equation

\[ T \propto \frac{\text{Volume}}{\text{Absorption}} \]  

(Eq. 13.1 of Kinsler et al.) relates the reverberation time (a measure of the decay time of an acoustic transient) to the losses (absorption) within an environment. The reverberation time \( T \) is usually defined as the time needed for an acoustic transient to be reduced to 60 dB below it’s peak value. The 60 dB dynamic range required of the measurement equipment is difficult to achieve without the use of large sound pressures or long averaging times. The techniques in this laboratory exercise will allow you to estimate reverberation times from measurements of much smaller dynamic range.

An exponential decay of reverberant sound energy. Prior to time = 0 a sound source has produced a diffuse-reverberant field of sound magnitude \( P=1 \). The sound source is turned off at \( t=0 \) and the reverberant field decays, such that

\[ P(t) = P(t = 0) e^{-t/\tau}, \]

with a time-constant \( \tau = 10 \) ms.

c. The balance of sound production and absorption in an echoic room.

Consider a sound source of intensity \( I_o \) in an echoic room. At some time \( (t = 0) \) the sound source is turned on. The power delivered to the room depends on some characteristic radiation area of the \( (S) \) of the sound source and the amount of energy that the source pumps into the room depends on the time that the source is on:

\[ \text{Sound power produced by the source, } \Pi_S = I_o S \]
Sound energy delivered to the room is the product of the power and the duration of the sound, $Wd = \Pi_S \Delta t$.

As the sound source remains on, two things happen, (1) the sound energy (or Power or Intensity) spreads around the room at a rate determined by the propagation velocity of sound, and (2) as the sound waves interact with the room walls, some fraction of the sound energy is reflected back into the room, following the laws of reflection, while the remaining fraction is absorbed by the walls.

The amount of sound power that is absorbed depends on the sound energy within the room and the absorbance with the room, $\Pi_A = W A$.

As the source remains on, a steady state is achieved, where the sound power absorbed equals the power produced by the source. The result is a relatively constant diffuse-field throughout the room in which the reverberant sound energy is uniformly dispersed.

The effect is analogous to a tub with an inflow of water and a drain. The outflow from the drain (the absorbance) depends on the level of the water in the tub (the energy level in the room). The steady state water level depends on a balance between the inflow and the outflow.
d. Direct and Diffuse field.

In general, the sound pressure at any point in space in a real echoic room can be characterized as the sum of waves propagating in all directions. In the one-dimensional case, we need only consider waves propagating in the positive and negative directions. In a three-dimensional reverberant environment, the number of possible wave directions is infinite. However, there are a few rules, which simplify the sound field. If the dimensions of the environment are large compared to sound wavelengths, there will be a region near the sound source where the sound field is dominated by the one-dimensional “direct field” that comes from the sound source. This region occurs because losses in air and the spread of sound power through the room reduce the magnitude of the reflected waves relative to the output wave near the source. As the stimulus wave propagates from the sound source its pressure amplitude is reduced by a factor of $1/r$ (Eqn 1) until the reflected waves dominate. If the sound stimulus continues for a long enough time, these reflected waves come from all directions and are all of similar average power density such that the sound pressure is relatively independent of position. This condition is called the diffuse field. The location where the direct and diffuse field meet is called the critical distance.