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Acoustics of Speech and Hearing

Notes on Acoustic Tubes and Vowels.*

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1 Introduction

Acoustic tubes have three important functions in the processes of speech and hearing: the ear canal conveys sound from the pinna to the eardrum, the cochlea performs a frequency analysis of sound, and the vocal tract generates speech.

![Figure 1: Schematic representation of the outer, middle, and inner ear.](image)

1.1 Ear Canal

Sound enters the ear canal (Fig. 1) is a relatively constant shape, air-filled tube that is roughly 2.6 cm long and 0.7 cm in diameter. The ear-canal is terminated by a relatively fixed tympanic membrane (ear drum), and so is often modelled as a tube that is open to the atmosphere at the pinna and terminated by a closure at the tympanic membrane. We shall see that the ear canal exhibits its lowest resonant frequency at roughly 3300 Hz, providing significant gain for frequencies within the spectral region of importance to speech.

1.2 Cochlea

After passing through the ossicles, sound is analyzed by the cochlea, a spiralled, fluid-filled, conical chamber in the temporal bone of the head. In humans the cochlea is roughly 3.5 cm long. Because the mechanical properties of the cochlea vary radically from one end to the other, it is often modelled as having a constant shape, but non-uniform mechanical properties along its length.
1.3 Vocal Tract

The vocal tract (Fig. 2), the organ used to produce speech, extends from the larynx to the lips and is typically a 17 cm long tube in adult men (shorter in women and children). Unlike the ear canal and the cochlea, whose shape is relatively fixed, the shape of the vocal tract is subject to volitional control that is exercised in the process of speech production.

These notes are primarily concerned with the processes by which vowel sounds are produced in the vocal tract. You are encouraged to visit

http://www.exploratorium.edu/exhibits/vocal_vowels/vocal_vowels.html

to discover how fairly simple, hollow, plastic models of the human vocal-tract can turn unintelligible squawks into recognizable vowels. Hopefully these notes will give you some insight into how these models were designed.

We begin by considering sound propagation in uniform tubes, including both their natural and driven behavior. We then model tubes as generalized two ports. We consider the important case of Helmholtz resonators. We consider nonuniform acoustic tubes the case of acoustic systems composed of two tube segments. We then deal with the effects of perturbations in the shape of a tube on the natural frequencies of these systems. We complete this discussion with an introduction to the production of vowels by acoustic tube systems in order to get insight into the production of speech in the vocal tract.
2 Sound Propagation in Tubes

Sound propagation in a rigid tube of uniform cross sectional area is a special type of plane wave propagation.

Figure 3: A lumped element approach to modelling a distributed system, such as an acoustic tube of uniform cross section

2.1 Circuit Model for a Tube

One way to study sound propagation in tubes is to model the tube as an interconnection of short segments, each of which has the properties of acoustic mass and acoustic compliance (Fig. 3).

If the segments (∆x) are sufficiently short, the pressure and volume velocity may be approximated as varying linearly within a segment so that each segment may be represented both as a small acoustic mass and a small compliance. If the length of the segment is ∆x, the acoustic mass of the segment is

\[ M∆x = \frac{\rho_0}{A}∆x, \]

where \( \rho_0 \) is the equilibrium mass density of the fluid in the tube, \( A \) is the area of a cross-section of the tube and \( M \) is the acoustic mass per unit length of the tube.
Similarly, the acoustic compliance of the segment is

\[ C \Delta x = \frac{A}{B_A} \Delta x = \frac{A}{\gamma P_0} \Delta x. \]

where \( B_A \) is the adiabatic compressibility of air, \( P_0 \) is the ambient pressure, \( \gamma \) is a ratio of specific heats, and \( C \) is the acoustic compliance per unit length of the tube.

### 2.2 Circuit Equations

The KUL and KPL equations for the circuit representing a segment of a rigid acoustic tube are

\[
\begin{align*}
    u(x - \Delta x, t) - u(x, t) &= C \Delta x \frac{\partial p(x, t)}{\partial t} \\
p(x, t) - p(x + \Delta x, t) &= M \Delta x \frac{\partial u(x, t)}{\partial t}
\end{align*}
\]

In the limit \( \Delta x \to 0 \), these equations become

\[
\begin{align*}
    -\frac{\partial u(x, t)}{\partial x} &= C \frac{\partial p(x, t)}{\partial t} \\
    -\frac{\partial p(x, t)}{\partial x} &= M \frac{\partial u(x, t)}{\partial t}
\end{align*}
\]
2.3 Tube Model of the Cochlea

This approach can also be used to develop some understanding of sound propagation within the cochlea. In this case (Fig. 4), the compliance element is replaced by a series connection of an acoustic mass, a compliance, and an acoustic resistance. All element values are made to be a function of cochlear position because the cross sectional area of the cochlea varies, causing the fluid mass to vary, and the mechanical properties (particularly the compliance) of the basilar membrane varies with position. The volume velocity through each $C$, $M$, $R$ branch is the effective neural stimulus.

![Diagram of distributed element model of sound propagation in a small segment of the cochlea. The fluid masses are represented by $N$ and the mechanical properties of the basilar membrane by $M$, $C$, and $R$ elements. All element values, as well as the volume velocity and pressure, are functions of position along the basilar membrane $x$.](image)

Figure 4: A distributed element model of sound propagation in a small segment of the cochlea. The fluid masses are represented by $N$ and the mechanical properties of the basilar membrane by $M$, $C$, and $R$ elements. All element values, as well as the volume velocity and pressure, are functions of position along the basilar membrane $x$. 
3 The Wave Equation

It is possible to eliminate the variable $p$ or $u$ from Eq. 1 and Eq. 1, yielding:

$$\frac{\partial^2 p(x, t)}{\partial x^2} = MC \frac{\partial^2 p(x, t)}{\partial t^2} = \frac{1}{c^2} \frac{\partial^2 p(x, t)}{\partial t^2}$$  \hspace{1cm} (1)

$$\frac{\partial^2 u(x, t)}{\partial x^2} = MC \frac{\partial^2 u(x, t)}{\partial t^2} = \frac{1}{c^2} \frac{\partial^2 u(x, t)}{\partial t^2}$$  \hspace{1cm} (2)

The general solution to these equations can be expressed in terms of pairs of forward and backward propagating waves, as in the case of plane waves that propagate in free space.

$$p(x, t) = p_+ (x - ct) + p_- (x + ct)$$

$$u(x, t) = u_+ (x - ct) - u_- (x + ct)$$

$$u(x, t) = \frac{1}{z_0} p_+ (x - ct) - \frac{1}{z_0} p_- (x + ct)$$

where

$$MC = \frac{\rho_0}{\gamma P_0} = \frac{1}{c^2}, \hspace{1cm} (3)$$

$c$ is the speed of sound in the medium. Similarly,

$$\sqrt{\frac{M}{C}} = \frac{\rho_0 c}{A} = z_0, \hspace{1cm} (4)$$

is the characteristic impedance of the tube. Subsequent work will make use of the fact that

$$c \times \frac{1}{z_0} = \frac{1}{M} \hspace{1cm} (5)$$

$$c \times z_0 = \frac{1}{C} \hspace{1cm} (6)$$

Note that the corresponding equations that govern sound propagation in plane waves describe pressure and particle velocity, rather than pressure and volume velocity. As a result the characteristic impedance term for plane waves was $\rho_0 c$ rather than $\rho_0 c/A$. As in the plane wave case, $c$ is inversely proportional to $\sqrt{\rho_0}$ and $z_0$ is directly proportional to $\sqrt{\rho_0}$. But, whereas $c$ is independent of $A$, $z_0$ is inversely proportional to $A$.

3.1 Sounds with Sinusoidal Time Dependence

In the special case of lossless tubes, it is also possible to express the solution to Eq. 1 and 2 as a product of a function of $x$ and a function of $t$. Since we are primarily interested in the case of sinusoidal time dependence, we choose the latter function as $e^{j\omega t}$
\[ p(t) = P(x)e^{j\omega t} \]
\[ u(t) = U(x)e^{j\omega t} \]

where \( P(x) \) and \( U(x) \) are the complex amplitudes of the pressure and volume velocity expressed as functions of position \( x \) within the tube.

Under these assumptions, Eq. 1 and 2 may be rewritten in the form

\[
\frac{dP(x)}{dx} = -j\omega MU(x) \tag{7}
\]
\[
\frac{dU(x)}{dx} = -j\omega CP(x) \tag{8}
\]
\[
\frac{d^2P(x)}{dx^2} = -\omega^2 MCP(x) = -k^2P(x) \tag{9}
\]
\[
\frac{d^2U(x)}{dx^2} = -\omega^2 MCU(x) = -k^2U(x) \tag{10}
\]

These equations are thus ordinary homogeneous differential equations and are readily solved. For example,

\[
\frac{d^2U(x)}{dx^2} = -\omega^2 MCU(x) = -k^2U(x),
\]

where,

\[ k = \omega\sqrt{MC} = \frac{\omega}{c}. \]

Eq. 10 has the well known solution

\[ U(x) = U_0 \sin (kx + \theta). \]

The values of the parameters \( U_0, \omega, \) and \( \theta \) are not specified by the differential equation. According to Eq. 8 the corresponding solution for \( P(x) \) is

\[
P(x) = \frac{1}{-j\omega C} \frac{dU(x)}{dx} = jz_0U_0 \cos (kx + \theta), \tag{11}
\]

so that

1. At each \( x \) the volume velocity and pressure waveforms are in time quadrature (90 degrees out of phase), as indicted by the \( j \) factor in the expression for \( P \),

2. \( U(x) \) and \( P(x) \) are in space quadrature as indicated by the \( \sin (kx + \theta) \) spatial dependence of \( U(x) \) and the \( \cos (kx + \theta) \) spatial dependence of \( P(x) \).
Similarly,
\[
\frac{d^2 P(x)}{dx^2} = -\omega^2 MC P(x) = -k^2 P(x),
\]
where, as above,
\[
k = \omega \sqrt{MC} = \frac{\omega}{c}.
\]
Eq. 9 has the well known solution
\[
P(x) = P_0 \sin(kx + \phi).
\]
The values of the parameters \(P_0\), \(\omega\), and \(\theta\) are not specified by the differential equation. According to Eq. 7 the corresponding solution for \(U(x)\) is
\[
U(x) = \frac{1}{-j\omega M} \frac{dP(x)}{dx} = j \frac{P_0}{z_0} \cos(kx + \phi),
\]
so that

1. The volume velocity and pressure waveforms are in time quadrature (90 degrees out of phase), as indicated by the \(j\) factor in the expression for \(U\).
2. \(P(x)\) and \(U(x)\) are in space quadrature as indicated by the \(\sin kx + \phi\) spatial dependence of \(P(x)\) and the \(\cos (kx + \phi)\) spatial dependence of \(U(x)\).
4 Natural Frequencies for Tubes

An important case of sound propagation in tubes is the case of no sources, the undriven or natural acoustic behavior of tubes. Since tubes are linear physical systems, it is reasonable to expect that sound can exist in the tube only at certain discrete frequencies, the natural frequencies of the tube. Unlike the case of systems composed of a finite number of lumped elements, which have a finite number of natural frequencies, distributed systems such as tubes have an infinite number of natural frequencies.

4.1 Case 1: Tube open at both ends.

A tube open to the atmosphere at both ends must satisfy the boundary conditions or end constraints that the sound pressure is zero at both ends of the tube (see Sec. 4.4 for a more complete discussion). The general form of the solution to equation Eq. 9, \( P(x) = P_0 \sin (kx + \phi) \), must satisfy the constraints

\[
P(0) = 0 \\
P(-L) = 0
\]

the first boundary condition requires that \( \sin \phi = 0 \) or \( \phi = 0 \)

and the second requires that \( \sin (-kL) = -\sin kL = 0 \). This is satisfied only if the product \( kL \) is an integer multiple of \( \pi \). Since \( L \) is a fixed constant, it is satisfied only if \( k \) is an integer multiple of \( \pi/L \)

\[
k_n = \frac{n \pi}{L} \quad \text{where } n = 0, 1, 2, \ldots .
\]

Each value of \( n \) corresponds to one of the natural frequencies of the acoustic system. In terms of \( f = \omega/(2\pi) \)

\[
f_n = \frac{n c}{2L} \quad \text{where } n = 0, 1, 2, \ldots . \quad (13)
\]

Thus

\[
\begin{align*}
P(x) &= P_0 \sin \left( n \pi \frac{x}{L} \right) \\
U(x) &= j \frac{P_0}{z_0} \cos \left( n \pi \frac{x}{L} \right)
\end{align*}
\]

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The presence of the factor $j$ indicates that the volume velocity phase leads the pressure phase by 90 degrees. The case $n = 0$ corresponds to steady flow through the tube with zero sound pressure. The distribution of pressure and volume velocity within the tube (often referred to as modes) for $n = 1, 2, 3$ are shown in Fig. 6.

Figure 6: The solid curve represents the amplitude of pressure ($|P(x)|$) and dotted curve represents the amplitude of volume velocity ($|U(x)|$) within a tube open at both ends. The three panels correspond to $n = 1, 2, 3$ from the bottom to the top. In each panel, the lower straight line corresponds to zero pressure and volume velocity. The curves are drawn under the assumption that $z_0 = 1$ acoustic ohm.

For all natural frequencies, note that the amplitude of sound pressure is zero at both ends of the tube because of the boundary conditions and the amplitude of the volume velocity is maximum at both ends of the tube. The pressure is said to have nodes at the ends of the tube and the volume velocity is said to have antinodes at the ends of the tube. That the maximum of volume velocity should occur at a node of pressure follows from Eq. 8. There is also a volume velocity node at the center of the tube. The pressure is maximal there (Eq. 7). Moreover there is a $\pi$ change in the phase of the volume velocity from the left half of the tube to the right.

For natural frequencies higher than the lowest, there are nodes of both pressure and volume velocity at locations within the tube. At each node of pressure (volume velocity) there is an antinode of volume velocity (pressure) and a $\pi$ change of phase across the node.

4.2 Case 2: Tube closed at both ends.

A tube closed to the atmosphere at both ends must satisfy the boundary constraints that the volume velocity is zero at both ends of the tube. The general form of the solution to
equation Eq. 9, \( U(x) = U_0 \sin (kx + \theta) \) must satisfy the constraints

\[
U(0) = 0 \quad U(-L) = 0
\]

the first boundary condition requires that \( \sin \theta = 0 \) or \( \theta = 0 \), and the second requires \( \sin -kL = -\sin kL = 0 \). This is satisfied only if the product \( kL \) is an integer multiple of \( \pi \). Since \( L \) is a fixed constant, it is satisfied only if \( k \) is an integer multiple of \( \pi/L \)

\[
k_n = n \frac{\pi}{L} \quad \text{where } n = 0, 1, 2, \ldots
\]

or in terms of \( f = \omega/(2\pi) \)

\[
f_n = n \frac{c}{2L} \quad \text{where } n = 0, 1, 2, \ldots
\]

The case \( n = 0 \) corresponds to constant pressure throughout the tube with no volume velocity. The distribution of pressure and volume velocity within the tube are shown in Fig. 8 for \( n = 1, 2, 3 \). Note that although the natural frequencies of a tube closed at both ends are the same as the natural frequencies of a tube open at both ends, the spatial distributions of pressures and volume velocities (modes) are different.

For all natural frequencies, note that the amplitude of volume velocity is zero at both ends of the tube because of the boundary conditions and the amplitude of the sound pressure is maximum at both ends of the tube. Volume velocity is said to have nodes at the ends of the tube and the sound pressure is said to have antinodes at the ends of the tube. That the maximum of sound pressure should occur at a node of volume velocity follows from Eq. 7. There is also a pressure node at the center of the tube. The volume velocity is maximal there (Eq. 8. Moreover there is a \( \pi \) change in the phase of the sound pressure from the left half of the tube to the right across the node.

For natural frequencies higher than the lowest, there are nodes of both pressure and volume velocity at locations within the tube. At each node of volume velocity (pressure) there is an antinode of pressure (volume velocity) and a \( \pi \) change of phase across the node.

4.3 Case 3: Tube closed at one end and open at the other.

A tube open to the atmosphere at one end and closed at the other must satisfy the boundary constraints that the sound pressure is zero at the open end of the tube and that the volume
Figure 8: The solid curve represents the amplitude of pressure ($P(x)$) and dotted curve represents the amplitude of volume velocity ($U(x)$) within a tube closed at both ends. The three panels correspond to $n = 1, 2, 3$ from the bottom to the top. In each panel, the lower straight line corresponds to zero pressure and volume velocity. The curves are drawn under the assumption that $z_0 = 1$ acoustic ohm.

Figure 9: A tube closed at one end and open at the other. Velocity is zero at the closed end. The general form of the solution to equation Eq. 9, $P(x) = P_0 \sin (kx + \phi)$ must satisfy the constraints

$$P(0) = 0, \quad U(-L) = 0$$

The first boundary condition requires that $\sin \theta = 0$ so $\theta = 0$. According to Eq. 7 when $U(x) = 0$, the derivative of $P$ with respect to $x$ must be zero. Thus the second constraint requires

$$0 = U(-L) = \frac{1}{j\omega M} \left. \frac{dP}{dx} \right|_{x=-L} = \frac{P_0}{Mc} \cos kL$$
Figure 10: The solid curve represents the amplitude of pressure ($|P(x)|$) and dotted curve represents the amplitude of volume velocity ($|U(x)|$) within a tube closed at one end and open at the other. The three panels correspond to $n = 1, 2,$ and 3 from the bottom to the top. In each panel, the lower straight line corresponds to zero pressure and volume velocity. The curves are drawn under the assumption that $z_0 = 1$ acoustic ohm.

or $\cos kL = 0$. This is satisfied only if the product $kL$ is an odd integer multiple of $\pi/2$. Since $L$ is a fixed constant, it is satisfied only if $k$ is an odd integer multiple of $\pi/2L$, so that

$$k_n = (2n + 1)\frac{\pi}{2L}, \text{ where } n = 0, 1, 2, \ldots,$$

or in terms of $f = \omega/(2\pi)$

$$f_n = (2n + 1)\frac{c}{4L} \quad \text{where } n = 0, 1, 2, \ldots \quad (18)$$

Note that the lowest natural frequency for the tube open at one end and closed at the other is one half that for the cases considered previously. The distribution of pressure and volume velocity within the tube are shown in Fig. 10 for the lowest three modes.

For all natural frequencies, note that the amplitude of volume velocity is zero at the closed end of the tube and the sound pressure is zero at the open end of the tube because of the boundary conditions. The amplitude of the sound pressure is maximum at the closed end of the tube. Volume velocity is said to have a node at the open end of the tube and the sound pressure is said to have an antinode there. Sound pressure is said to have a node at the closed end of the tube and the volume is said to have an antinode there. That the maximum of sound pressure should occur at a node of volume velocity follows from Eq. 7. That the maximum of volume velocity should occur at a node of pressure follows from Eq. 8.
For natural frequencies higher than the lowest, there are nodes of both pressure and volume velocity at locations within the tube. At each node of volume velocity (pressure) there is an antinode of pressure (volume velocity) and a $\pi$ change of phase across the node.

Note the relation between the distributions of pressure and volume velocity between the tube that is closed at one end and open at the other and the tubes that are either open or closed at both ends. At the first natural frequency, the tube that is open at both ends has a volume velocity node at its center. Thus its first natural frequency is the same as that of a tube closed at one end and open at the other but half as long. Many such relations can be found between the natural frequencies of these three basic types of tubes.

4.4 Correction of length for end effects.

The assumption that $P = 0$ at an end of a tube open to the atmosphere ignores the fact that sound is radiated from the tube and can be detected in the atmosphere. The air at the end of the tube acts somewhat like a piston or vibrating sphere and encounters a radiation impedance. As a result the pressure at the end of the tube is not precisely zero. The effect is to change each natural frequency (see Sec. 6). For the lowest natural frequency, this change is as if the length of the tube were greater than its physical length. For a simple round tube, this increase is roughly $0.03D$, where $D$ is the diameter of the the tube, for each end open to the atmosphere. Modelling this effect will be discussed further in Sec. 6.1 and Sec. 12.
5 System Functions for Tubes

Like other acoustic systems, tubes can be connected to volume velocity or pressure sources. At such connections, tubes present impedances/admittances to the source. There are also transfer functions that relate the pressure or volume velocity of the source to pressures or volume velocities at other points in the tube.

Figure 11: A tube open at one end \((x = 0)\) and driven by a volume velocity source at the other \((x = -L)\).

5.1 Tube Open at One End.

Consider the tube in Fig. 11. It differs from the tube in Fig. 6 in that the non-open end is connected to a volume velocity source \(U_S\), which could be a piston. This causes the boundary condition at \(x = -L\) to be \(U(-L) = U_S\) without affecting the boundary condition at \(x = 0\), \(P(0) = 0\).

If the amplitude of the volume velocity in the tube is expressed as

\[
U(x) = U_0 \cos (kx + \theta)
\]

then making use of Eq. 6 and 8, the amplitude of the pressure in the tube is

\[
P(x) = -\frac{1}{j\omega C} \frac{dU}{dx} = -j\frac{\omega}{\omega C} U_0 \sin (kx + \theta) = -jz_0 U_0 \sin (kx + \theta)
\]

The boundary condition \(P(0) = 0\) requires that \(\sin \theta = 0\) so that \(\theta = 0\). The boundary condition \(U(-L) = U_S\) requires that \(U_0 \cos kL = U_S\). The impedance that the tube presents to the volume velocity source is

\[
Z(-L) = \frac{P(-L)}{U_S} = -jz_0 \frac{U_0 \sin (-kL)}{U_0 \cos kL} = jz_0 \frac{\sin kL}{\cos kL} = jz_0 \tan kL.
\]

This impedance is purely imaginary because no possible sources of energy loss, e.g., viscosity, were included in the model of the tube.

Note that for small \(\omega\), making use of Eq. 5,

\[
Z(-L) = jz_0 \tan kL = jz_0 \tan \frac{\omega L}{c} = jz_0 \frac{\omega L}{c} = j\omega ML
\]
where $ML$ is the total acoustic mass of the tube.

The normalized impedance $Z' = Z/z_0$ may be expressed in terms of the normalized frequency $\Omega = \omega/(c/L)$

$$Z'(-L) = \frac{Z(-L)}{z_0} = j\tan \Omega = j\frac{\sin \Omega}{\cos \Omega}$$

At all frequencies, since $Z'$ has no real part, i.e., $Z'(-L) = jX'$. The dependence of the normalized reactance $X' = \tan \Omega$ on normalized frequency is shown in Fig. 12.

![Diagram](image)

**Figure 12:** The solid curve represents the the dependence of the normalized reactance $X' = X/z_0$ on normalized frequency $\Omega = \omega/\omega_0$, where $\omega_0 = c/L$, for a tube open at one end and driven by a volume velocity source at the other. The dashed vertical lines mark the frequencies of the (imaginary-valued) poles of $Z(-L)$.

It is of some interest to determine the volume velocity transfer ratio for this tube

$$H(j\omega) = \frac{U(0)}{U_S} = \frac{U_0}{U_0 \cos kL} = \frac{1}{\cos kL} = \frac{1}{\cos \frac{\omega L}{c}} = \frac{1}{\cos \Omega}$$

The dependence of the magnitude of the volume velocity transfer ratio $|H(-L)|$ on frequency is shown in Fig. 13.

Note that both the impedance $Z$ and the volume velocity transfer ratio $H$ have (imaginary valued) poles at those values of $\omega$ for which $\cos (\omega L/c) = 0$, i.e. for which

$$\omega_n = \frac{\pi c}{2L}, \frac{3\pi c}{2L}, \frac{5\pi c}{2L}, \ldots$$

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Figure 13: The solid curve represents the dependence of the magnitude of the volume velocity transfer ratio $|H|$ on normalized frequency $\Omega$ where $\Omega = \omega/\omega_0$ and $\omega_0 = c/L$ for a tube open at one end and driven by a volume velocity source at the other. The dashed vertical lines mark the frequencies of the (purely imaginary) poles of $H$.

or in terms of $f = \omega/2\pi$

$$f_n = \frac{c}{4L}, \frac{3c}{4L}, \frac{5c}{4L}, \ldots$$

These are just the natural frequencies of a tube of length $L$ that is open at one end and closed at the other. This equivalence corresponds to the fact that when a volume velocity source has zero intrinsic value, it cannot be distinguished from a rigid termination.

This illustrates a more general fact. The natural frequencies of the tube system are the poles of all the system functions of the system. It should be noted that the poles of the system functions occur in complex conjugate pairs, so

$$\frac{1}{\cos kL} = \frac{1}{\cos \omega L/c} = \frac{S_1 S_1^*}{(j\omega - S_1)(j\omega - S_1^*)} \times \frac{S_2 S_2^*}{(j\omega - S_2)(j\omega - S_2^*)} \times \cdots$$

where

$$S_n = j(2n - 1) \frac{c}{4L}$$

or on simplifying

$$\frac{1}{\cos kL} = \frac{1}{\cos \omega L/c} = \prod_{n=1}^{\infty} \frac{1}{1 - (\omega/\omega_n)^2}$$

where

$$\omega_n = (2n - 1) \frac{c}{4L} \text{ for } n = 1, 2, \ldots$$
Figure 14: Distribution of volume velocity (dashed curves) and pressure (solid curves) within a tube open at the right end and driven by a volume velocity source at the left end. The five panels correspond to the normalized frequencies $\Omega = 0.25\pi, 0.425\pi, 0.575\pi, 0.925\pi, 1.075\pi$, increasing from the bottom to the top. In each panel, the lower straight line corresponds to zero pressure/volume velocity. The volume velocity at the left end of the tube $(|U_S|)$ is the same in all panels. The curves for pressure correspond to $z_0 = 1$ acoustic ohm. Note: the relative amplitudes of pressure and volume velocity are to scale.

It is also of interest to examine the distribution of pressure and volume velocity within a tube that is open at one end and driven by a volume velocity source at the other. The equations for volume velocity and pressure within the tube may be written as

$$U(X) = \frac{\cos kx}{\cos kL} U_S$$
$$P(X) = jz_0 \frac{\sin kx}{\cos kL} U_S$$

These equations may be rewritten in terms of the normalized frequency $\Omega = \omega/\omega_0$ where $\omega_0 = c/L$, normalized position along the length of the tube $\chi = x/L$, where $-1 \leq \chi \leq 0$, and characteristic impedance $z_0$ as

$$U(\chi) = \frac{\cos \chi \Omega}{\cos \Omega} U_S$$
$$P(\chi) = -jz_0 \frac{\sin \chi \Omega}{\cos \Omega} U_S$$
Fig. 14 plots the distribution of $|U(\chi)|$ and $|P(\chi)|$ for several values of $\Omega$, including a relatively small value, two values near $\Omega = \pi/2$ and two values near $\Omega = \pi$.

- At low frequencies, $U(\chi) \approx U_S$ throughout the tube\(^1\) and that $|P(\chi)|$ decreases roughly linearly from a maximum value $z_0 U_S \tan \Omega \approx ML \omega U_S$ at the position of the volume velocity source to zero at the open end of the tube.

- As $\Omega$ increases, but is still less than $\pi/2$, the maximum amplitude of $P(\chi)$ continues to occur at the position of the volume velocity source, $\chi = -1$. The maximum amplitude of $U(\chi)$, occurs at the open end of the tube, $\chi = 0$. The value of the maximum amplitude of $P(\chi)$ increases, as does the value of maximum amplitude of $U(\chi)$.

- As $\Omega$ increases beyond $\pi/2$, a node appears in $U(\chi)$ just to the right of $\chi = -1$. This node occurs where the spatial derivative of $P(\chi)$ is zero. Only the volume velocity to the left of this node is in phase with the source, that to the right of the node is out of phase (by 180 deg or $\pi$ radians) with the source.

- As $\Omega$ continues to increase, but is still less than $\pi$, the position of the node moves toward the open end of the tube and the amplitude of the pressure at the volume velocity source decreases, approaching 0 when $\Omega = \pi$. Increasing $\Omega$ beyond $\pi$ introduces a node in $P(\chi)$ just to the right of $\chi = -1$.

5.2 Tube Closed at One End.

![Figure 15: A tube closed at one end ($x = 0$) and driven by a volume velocity source at the other ($x = -L$).](image)

Figure 15: A tube closed at one end ($x = 0$) and driven by a volume velocity source at the other ($x = -L$).

Consider the tube in Fig. 15. It differs from the tube in Fig. 10 in that the non-closed end is connected to a volume velocity source $U_S$, which could be a piston. This causes the boundary condition at $x = -L$ to be $U(-L) = U_S$ without affecting the boundary condition at $x = 0$, $U(0) = 0$. If the amplitude of the volume velocity in the tube is expressed as

$$U(x) = U_0 \sin (kx + \theta)$$

then making use of Eq. 6 and 8, the amplitude of the pressure in the tube is

$$P(x) = jz_0 U_0 \cos (kx + \theta)$$  \(20\)

\(^1\)This is slightly inaccurate. Except for $\Omega = 0$ the the volume velocity at the open end of the tube is larger than $U_S$. 19
The boundary condition $U(0) = 0$ requires that $\sin \theta = 0$ so that $\theta = 0$. The boundary condition $U(-L) = U_S$ requires that $-U_0 \sin kL = U_S$. The impedance of the tube seen by the volume velocity source is

$$Z(-L) = \frac{P(-L)}{U_S} = -j z_0 \cot kL \quad (21)$$

Note that for small $\omega$, making use of Eq. 6,

$$Z(-L) = -j z_0 \cot kL = -j z_0 \frac{\omega L}{c} = -j z_0 \frac{c}{\omega L} = \frac{1}{j \omega CL}$$

where $CL$ is the total acoustic compliance of the tube.

This impedance is purely imaginary because no possible sources of energy loss, e.g., viscosity, were included in the model of the tube. The normalized impedance $Z' = Z/z_0$ may be expressed in terms of the normalized frequency $\Omega = \omega/(c/L)$

$$Z'(-L) = \frac{Z(-L)}{z_0} = -j \cot \Omega$$

At all frequencies, this impedance has no real part, being a purely imaginary number, i.e. $Z'(-L) = jX'$. The dependence of the normalized reactance $X' = \tan \Omega$ on normalized frequency is shown in Fig. 16.

Figure 16: The solid curve represents the dependence of the normalized reactance $X/z_0$ on normalized frequency $\Omega$ where $\Omega = \omega/\omega_0$ and $\omega_0 = c/L$, for a tube closed at one end and driven by a volume velocity source at the other. The dashed vertical lines mark the frequencies of the (imaginary-valued) poles of $Z(-L)$. 

20
It is of some interest to determine the transfer impedance relating the pressure at the closed end of the tube to the volume velocity of the source for this tube

\[
Z_T(j\omega) = \frac{P(0)}{U_S} = -\frac{jz_0}{\sin kL} = -jz_0 \frac{1}{\sin \Omega}
\]

The dependence of the magnitude of the normalized transfer impedance \( |Z_T(-L)/z_0| \) on frequency is shown in Fig. 17

![Transfer Function Magnitude vs Normalized Frequency](image)

**Figure 17:** The solid curve represents the dependence of the magnitude of the normalized transfer impedance \( |Z_T(z_0)\) on normalized frequency \( \Omega \) where \( \Omega = \omega/(c/L) \). for a tube closed at one end and driven by a volume velocity source at the other.

Note that both the impedance \( Z \) and the transfer impedance \( Z_T \) have poles at those values of \( \omega \) for which \( \sin (\omega L/c) = 0 \), i.e. frequencies for which \( \omega_n = n\pi c/L \), where \( n = 0, 1, 2, \ldots \)
or in terms of \( f = \omega/2\pi \)

\[
f_n = n\frac{c}{2L} \quad n = 1, 2, 3, \ldots
\]

These are just the natural frequencies of a tube of length \( L \) that is closed at both ends. This equivalence corresponds to the fact that when a volume velocity source has zero intrinsic value, it cannot be distinguished from a rigid termination.

This illustrates a more general fact. The natural frequencies of the tube system are the poles of all the system functions of the system. It should be noted that the poles of the system functions occur (except for the pole at zero) in complex conjugat pairs, so

\[
\frac{1}{\sin kL} = \frac{1}{\sin \omega L/c} = \frac{1}{j\omega} \times \frac{S_1S_1^*}{(j\omega - S_1)(j\omega - S_1^*)} \times \frac{S_2S_2^*}{(j\omega - S_2)(j\omega - S_2^*)} \times \ldots
\]
where
\[ S_n = jm \frac{c}{2L} \]

or on simplifying
\[ \frac{1}{\sin kL} = \frac{1}{\sin \omega L/c} = \frac{1}{j\omega} \times \prod_{m=1}^{\infty} \frac{1}{1 - (\omega/\omega_m)^2} \]

where
\[ \omega_m = \frac{m c}{2L} \text{ for } m = 1, 2, \ldots \]

It is also of interest to examine the distribution of pressure and volume velocity within a tube that is closed at one end and driven by a volume velocity source at the other. The equations for volume velocity and pressure within the tube may be written as
\[ U(X) = \frac{\sin kx}{\sin kL} U_S \]
\[ P(X) = -jz_0 \frac{\cos kx}{\sin kL} U_S \]

These equations may be rewritten in terms of the normalized frequency \( \Omega = \omega/\omega_0 \) where \( \omega_0 = c/L \), normalized position along the length of the tube \( \chi = x/L \), where \(-1 \leq \chi \leq 0\), and characteristic impedance \( z_0 \) as
\[ U(\chi) = -U_S \frac{\sin \chi \Omega}{\sin \Omega} \quad (22) \]
\[ P(\chi) = -jz_0 U_S \frac{\cos \chi \Omega}{\sin \Omega} \quad (23) \]

Fig. 18 plots the distribution of \( |U(\chi)| \) and \( |P(\chi)| \) for several values of \( \Omega \), including a relatively small value, two values near \( \Omega = \pi/2 \) and two values near \( \Omega = \pi \).

- At low frequencies, \( |P(\chi)| = z_0 U_S \cot \Omega \approx \frac{U_S}{kL} \) throughout the tube\(^2\) and \( U(\chi) \) decreases roughly linearly from a maximum value \( (U_S) \) at the position of the volume velocity source to zero at the open end of the tube.

- As \( \Omega \) increases, but is still less than \( \pi/2 \), the maximum amplitude of \( P(\chi) \), which occurs at the volume velocity source, \( \chi = -1 \), decreases. The maximum amplitude of \( U(\chi) \), which occurs at the volume velocity source, \( \chi = -1 \) remains the same.

- As \( \Omega \) increases beyond \( \pi/2 \), a node appears in \( P(\chi) \) just to the right of \( \chi = -1 \). This node occurs where the spatial derivative of \( U(\chi) \) is zero. At this position the amplitude of \( U(\chi) \) is greater than \( U_S \). The pressure at the closed end is now 180 deg out of phase with that at the pressure source.

\(^2\)This is slightly inaccurate. Except for \( \Omega = 0 \) the pressure at the open end of the tube is larger than \( z_0 U_S \).
• As Ω continues to increase, but is still less than π, the position of the pressure node moves toward the open end of the tube and the amplitude of the pressure at the volume velocity source increases. Increasing Ω beyond π introduces a node in $U(\chi)$ just to the right of $\chi = -1$.

Figure 18: Distribution of volume velocity (dashed curves) and pressure (solid curves) within a tube closed at the right end and driven by a volume velocity source at the left end. The five panels correspond to the normalized frequencies $\Omega = 0.25\pi, 0.425\pi, 0.575\pi, 0.925\pi, 1.075\pi$, increasing from the bottom to the top. In each panel, the lower straight line corresponds to zero pressure/volume velocity. The volume velocity at the left end of the tube ($|U_S|$) is the same in all panels. The curves for pressure correspond to the value $z_0 = 1$ acoustic ohm. Note: the relative amplitudes of pressure and volume velocity are to scale.
6 Tubes as Two-Ports.

Treating the ends of tubes as either closed or open to the atmosphere, is a special instance of the more general case in which both ends are connected to one-port acoustic circuits. In this sense the closed end of a tube may be regarded as an acoustic “open circuit” \( U = 0 \) and the open end of a tube as an acoustic “short circuit” \( P = 0 \). The acoustic tube itself may be treated as a generalized acoustic two-port.

Consider the tube in Fig. 19 in which two independent volume velocity sources \( U_1 \) and \( U_2 \), each of which has an \( e^{st} \) time dependence, are connected to opposite ends of the tube\(^3\). We represent the tube as a two-port in which the volume velocities at the ports are the independent variables and the pressures at the ports are the dependent variables:

\[
\begin{align*}
U(-L) &= U_1 \\
U(0) &= U_2 \\
x &= -L \\
\end{align*}
\]

Figure 19: A tube that is connected to two independent volume velocity sources, \( U_1 \) and \( U_2 \).

\[
\begin{align*}
P_1 &= Z_{11} U_1 + Z_{12} U_2 \quad (24) \\
P_2 &= Z_{21} U_1 + Z_{22} U_2 \quad (25)
\end{align*}
\]

Since

\[ Z_{11} = \left. \frac{P_1}{U_1} \right|_{U_2=0} \]

\( Z_{11} \) is just the impedance seen by a volume velocity source connected to a tube of length \( L \) that is closed at the other end, shown in Sec. 5.2 to be

\[ Z_{11} = -jz_0 \cot kL \]

By symmetry, \( Z_{22} \), defined as

\[ Z_{22} = \left. \frac{P_2}{U_2} \right|_{U_1=0} \]

equals \( Z_{11} \)

\[ Z_{22} = -jz_0 \cot kL. \]

The impedance \( Z_{21} \), defined as

\[ Z_{21} = \left. \frac{P_3}{U_1} \right|_{U_2=0} \]

\(^3\)We could also have considered drives composed of pressure sources or mixed pressure and volume velocity sources.
is equal to the transfer impedance \( Z_T \) that relates the pressure at the closed end of a tube of length \( L \) to the volume velocity at the other end

\[
Z_{21} = Z_T = \frac{-jz_0}{\sin kL}.
\]

By symmetry (and reciprocity) \( Z_{21} = Z_{12} \).

A circuit model for the two port described by Eq. 24 and 25 is shown in Fig. 20.

\[\text{Figure 20: Circuit model for the two-port described by Eq. 24 and 25.}\]

6.1 Example

\[\text{Figure 21: A tube that is closed at one end, and connected to a lumped acoustic mass } M_R \text{ at the other.}\]

As mentioned in Sec. 4.4 the pressure at an end of a tube that is open to the atmosphere is only approximately zero because sound is radiated from the tube. The air at the end of the tube encounters a radiation impedance \( Z_R \) that we will approximate as an acoustic mass\(^4\) as shown in Fig. 21. The presence of this mass \( M_R \) alters the system function relating \( P(0) \) to \( U_S \) when compared to a tube such as that in Fig. 9.

\(4\)We ignore the radiation resistance for simplicity.

Using a two-port model for the tube, a circuit model for this acoustic system is constructed in Fig. 22.

Rather than adopt an algebraic approach to determine the effect of \( Z_M \) on the volume velocity at node \( C \) in this circuit, we will attempt to determine the poles and zeroes of
the system function \( G(j\omega) = U_2/U_1 \). Clearly \( G \) must be zero when \( Z_{12} \) is zero. However \( Z_{12} = -jz_0/\sin kL \) has no zeroes; \( G \) is zero only when \( Z_{22} - Z_{12} = 2jz_0 \sin^2 (kL/2)/\sin kL \) is infinite, i.e. when \( kL = \pi, 3\pi, \ldots \).

The poles of \( G \) are the values of \( \omega \) for which it is possible to have nonzero pressures and volume velocities in the circuit of Fig. 23. There can be no volume velocity through the leftmost branch, but it is possible for volume velocity to exist in \( Z_{12} \), \( Z_{22} - Z_{12} \), and \( Z_R \), provided KPL is satisfied. Since the volume velocity through these three elements is the same, KPL is satisfied if the sum of the impedances around the loop is zero.

\[
Z_R + Z_{12} + (Z_{22} - Z_{12}) = Z_R + Z_{22} = 0
\]

or

\[
-jz_0 \cot kL + j\omega M_R = 0
\]

\[
\cot \left( \frac{\omega}{c/L} \right) = \frac{\omega cM_R}{c/L z_0 L}
\]

Eq. 26 could also be obtained by reasoning that the impedance seen by the acoustic mass when \( U_S = 0 \) must be the negative of the impedance of the acoustic mass at the natural
frequencies of the acoustic system. The roots of Eq. 26 correspond to those values of \( \omega = (c/L)\Omega \) for which

\[
\cot \Omega = \frac{M_R}{LM} \Omega = \mu \Omega,
\]

where \( \mu = M_R/ML \) is the ratio of the lumped element acoustic mass to the acoustic mass of the medium in the tube. However Eq. 27 is transcendental and cannot be solved by algebraic techniques. The values of \( \Omega \) that satisfy Eq. 27 can be determined graphically, as shown in Fig. 24.

Note that the natural frequencies of the tube in Fig. 9 are the values of \( k \) for which \( \cos kL = 0 \). These are the zeroes of \( Z_{22} \). Adding \( Z_R \) at the open end of the tube lowers each of the natural frequencies towards the natural frequencies of the tube in Fig. 7. The shift is small for the lowest natural frequencies, where the impedance \( Z_R \) is relatively low, but nearly complete for the higher natural frequencies, where the lumped acoustic mass behaves like a rigid wall.

Figure 24: The solid curve represents the the dependence of \( \cot \Omega \), the negative of the normalized reactance \( X/z_0 \) of the tube on normalized frequency \( \Omega = \omega/(c/L) \). the dashed straight line represents the reactance of the lumped element acoustic mass, and corresponds to the case \( \mu = 1 \) or \( M_R = ML \). The frequencies of the filled circles are the natural frequencies of the acoustic system.
Helmholtz Resonators

Helmholtz Resonators are composed of two tube segments (Fig. 25) with the larger area one closed to the outside and the smaller area one open to the outside.

One might think that the natural frequencies of this two-tube acoustic system could be analyzed by combining the natural frequencies of a tube of length $L_B$ that is closed at both ends

$$B_i = i \frac{c}{2L_B}$$

with the natural frequencies of a tube of length $L_F$ that is open at both ends

$$F_j = j \frac{c}{2L_F}$$

This casual analysis misses the important Helmholtz resonance, which is generally lower in frequency than either $B_1 = c/2L_B$ or $F_1 = c/2L_F$.

7.1 Short Tube Open at One End

If the length $L$ of the tube open at one end and driven by a volume velocity source at the other is sufficiently small that $\sin kL \approx kL$ and $\cos kL \approx 1$, then the impedance connected to the source can be approximated as

$$Z(-L) = jz_0kL$$

$$= j\frac{M}{C}\omega\sqrt{MC}L$$

$$= j\omega(ML)$$
Since $M$ is the acoustic mass per unit length and $L$ is the length of the tube, $ML = M_A$, the acoustic mass of the medium in the tube, and $Z(-L) = j\omega M_A$. This is same impedance presented by an acoustic mass $M_A$. It is also possible to normalize the impedance relative to $z_0$ and express it in terms of a normalized frequency $\Omega = \omega/(c/L)$

$$Z'(-L) = \frac{Z(-L)}{z_0} = \frac{j\omega ML}{\sqrt{M/C}} = \frac{j\omega L}{c} = j\Omega$$

A comparison of the frequency dependence of the normalized reactance of a tube with acoustic mass per unit length $M$ and length $l$ and of an acoustic mass $M_A = ML$ is shown in Fig. 26. Note that the reactance of the acoustic mass approximates that of the tube for small values of $\Omega$, e.g. $\Omega < 0.25$. When $\Omega = 0.25$, the reactance of the tube is roughly 1% larger than the reactance of the equivalent mass.

The volume velocity transfer function for a short tube is $H(j\omega) = 1$.

![Figure 26](image)

Figure 26: The solid curve represents the the dependence of the normalized reactance $X/z_0$ of an acoustic tube with acoustic mass per unit length $M$ and length $L$ on normalized frequency $\Omega$, where $\Omega = \omega/(c/L)$. The dashed curve represents the normalized reactance of an acoustic mass $M_A = ML$.

### 7.2 Short Tube Closed at One End

If the length $L$ of the tube closed at one end and driven by a volume velocity source at the other is sufficiently small that $\sin kL \approx kL$ and $\cos kL \approx 1$, then the impedance connected to the source can be approximated as

$$Z(-L) = -jz_0\frac{z_0}{kL} = -j\frac{\sqrt{M}}{C} \frac{1}{\omega L \sqrt{MC}} = \frac{1}{j\omega (CL)}.$$
Since $C$ is the acoustic compliance per unit length and $L$ is the length of the tube, $CL = C_A$, the acoustic compliance of the medium in the tube, and $Z(-L) = -j\omega C_A$. This is same impedance presented by an acoustic compliance $C_A$. It is also possible to normalize the impedance relative to $z_0$ and express it in terms of a normalized frequency $\Omega = \omega/(c/L)$

$$\frac{Z(-L)}{z_0} = \frac{1}{j\omega LC} \sqrt{\frac{C}{M}} = \frac{1}{j\omega L} \frac{1}{\sqrt{MC}} = \frac{1}{j\Omega}$$

(28)

A comparison of the frequency dependence of the reactance of a tube with acoustic compliance per unit length $C$ and length $L$ and of an acoustic compliance $C_A = CL$ is shown in Fig. 27. Note that the reactance of the acoustic mass approximates that of the tube for small values of $\Omega$, e.g. $\Omega < 0.25$. When $\Omega = 0.25$, the magnitude of the reactance of the tube is roughly 1% smaller than the magnitude of the reactance of the equivalent compliance.

![Normalized Reactance vs Normalized Frequency]

Figure 27: The solid curve represents the the dependence of the normalized reactance $X/z_0$ of an acoustic tube with acoustic compliance per unit length $C$ and length $L$ on normalized frequency $\omega/\omega_0$, where $\omega_0 = c/L$. The dashed curve represents the normalized reactance of an acoustic compliance $C_A = CL$. Note that while the reactance of the acoustic compliance is always negative, the reactance of the tube becomes positive for high enough frequency.

Since the pressure in a short tube is uniform, $Z_T(j\omega) = Z(j\omega)$.

### 7.3 Helmholtz Natural Frequency

In order to determine the natural frequency of the Helmholtz resonance, we start by assuming that at the this frequency ($f_0$) the length $L$ of each segment of the tube system is sufficiently small that

$$\sin \frac{2\pi f_0 L}{c} \approx \frac{2\pi f_0 L}{c}$$
and that
\[ \cos \frac{2\pi f_0 L}{c} \approx 1. \]
In this case, the back segment can be modelled as a simple acoustic compliance
\[ C = \frac{A_B L_B}{\rho_0 c^2} \]
and the front segment can be modelled as a simple acoustic mass
\[ M = \frac{\rho_0 L_F}{A_F} \]
the two elements being connected in series or in parallel.

The natural frequency, the Helmholtz Resonance, \( f_H \) of such a structure (according to the results of Sec. 8.3 of Notes on Circuits)
\[ f_H = \frac{1}{2\pi} \frac{1}{\sqrt{MC}} = \frac{c}{2\pi} \sqrt{\frac{A_F/A_B}{L_F L_B}} \]

For example, if \( L_F = 12 \) cm, \( L_B = 5 \) cm, \( A_F = 1 \) sq cm, \( A_B = 10 \) sq cm, then the lowest of the \( B_i \) and \( F_j \) is roughly 1400 Hz, whereas the Helmholtz resonance is roughly 223 Hz. At 223 Hz, \( X = 2\pi 223 \times 12/34300 = 0.49 \) whereas \( \sin X = 0.47 \), a difference of roughly 4%, and \( \cos X = 0.88 \).
8 Systems Composed of Two Tube Segments.

Because of their relevance to speech production, we consider systems consisting of two tubular parts: a closed back part of area $A_B$ and length $L_B$ and an open front part of area $A_F$ and length $L_F$ (Fig. 28). However, the principles we will develop apply to two tube systems in which the segments are both closed and both open.

The vowel /a/ in father has a vocal tract shape similar to the configuration of Fig. 28 when $A_B/A_F < 1$. On the other hand, when $A_B/A_F \gg 1$, the system can be treated as a Helmholtz resonator (Sec. 7). The vowel /u/ in boot has a vocal tract shape that corresponds roughly to this configuration. When $A_B/A_F = 1$, the system is identical to a uniform tube closed at one end (Sec. 4.3 and 5.2).

8.1 Limiting Cases

Two important limiting cases occur when $A_B/A_F \gg 1$ and when $A_B/A_F \ll 1$. In these cases the two sections of the two-tube system are nearly uncoupled.

The case $A_B/A_F \ll 1$.

The tube configuration in this case is shown in Fig. 29. Because $A_B/A_F \ll 1$, very little volume velocity leaves the rear of the front tube, so that tube is effectively closed at one end and open at the other. Similarly, the rear tube sees a large opening at its front, and so the rear tube is nearly also closed at one end and open at the other.

In this case, the separate natural frequencies of the front and back segments of the two-tube system are

$$B_i = \frac{2i - 1}{4} \frac{c}{L_B}$$  \hspace{1cm} (29)
Figure 29: Illustrating the case of two nearly uncoupled tubes with $A_B/A_F \ll 1$.

Front Segment:

$$F_j = \frac{2j-1}{4} \frac{c}{L_F} = \frac{2j-1}{4} \frac{c}{L - L_B}$$  \hspace{1cm} (30)

for $i, j = 1, 2, \ldots$, where $L = L_B + L_F$ is the total length of the tube.

The natural frequencies of the system are (very nearly) the union of the $B_i$ and the $F_j$. Note that the area ratio does not enter into these expressions because it was assumed that $A_B/A_F \rightarrow 0$.

Fig. 30 illustrates these natural frequencies of a tube of length $L = L_B + L_F = 17$ as a function of $L_B$ cm in the limiting case $A_B/A_F \rightarrow 0$.

Figure 30: Illustrating the approximate solution to Eq. 34 in the case $L_B = 5$ cm, $L_F = 12$ cm, $A_B/A_F \rightarrow 0$. The rising curves represent the $F_j$ and the falling curves the $B_i$. Filled squares axis indicate the approximate natural frequencies of the two-tube system.
Figure 31: Illustrating the case of two nearly uncoupled tubes with $A_B/A_F \gg 1$.

The case $A_B/A_F \gg 1$.

The tube configuration in this case is shown in Fig. 31. Because $A_B/A_F \gg 1$, very little volume velocity enters the rear tube from the front tube, so that tube is effectively closed at both ends. Similarly, the front tube sees a large opening at its rear, and so the front tube is nearly open at both ends.

In this case, the separate natural frequencies of the front and back segments of the two-tube system are

\[
\begin{align*}
\text{Back Segment:} & \quad B_i = i \frac{c}{2 L_B} \\
\text{Front Segment:} & \quad F_j = j \frac{c}{2 L_F} = j \frac{c}{2 L - L_B}
\end{align*}
\]

for $i, j = 1, 2, \ldots$, where $L = L_B + L_F$ is the total length of the tube.

The Helmholtz Resonance

As can be seen in Fig. 33, when $A_B/A_F \gg 1$ the natural frequencies of the two tube system occurs when

\[Z_B + Z_F = 0\]

where

\[Z_B = -j \frac{\rho_0 c}{A_B} \cot k L_B \approx -j \frac{\rho_0 c}{A_B} \frac{1}{k L_B}\]

\[Z_F = j \frac{\rho_0 c}{A_F} \tan k L_F \approx j \frac{\rho_0 c}{A_F} k L_F\]

This occurs when

\[k_H^2 = \frac{A_F}{L_F L_B A_B}\]

or when

\[F_H = \frac{c}{2 \pi} \sqrt{\frac{A_F}{A_B}}\]

\[(33)\]
The natural frequencies of the system are (very nearly) the union of the $B_i$ and the $F_j$ and the Helmholtz resonance. Note that the area ratio does not enter into the expressions for the $B_i$ and the $F_j$ because it was assumed that $A_B/A_F \to 0$, whereas it does enter into the expression for $F_H$. 
8.2 The General Case.

In the general case, we can determine the natural frequencies of the system of Fig. 28 by assuming that a source $U_S$ excites the system at the point where the two tube segments join (Fig. 32). The impedance seen by this source is clearly the parallel combination of $Z_B$ and $Z_F$:

$$\frac{P}{U_S} = Z = \frac{Z_B Z_F}{Z_B + Z_F}$$

The natural frequencies of the system of Fig. 28 are just the poles of this system function, or the zeroes of the denominator:

$$Z_B + Z_F = 0$$

When $Z_B$ represents the impedance of a closed tube of length $L_B$ and area $A_B$:

$$Z_B = -j \frac{\rho_0 c}{A_B} \cot k L_B$$

and $Z_F$ represents the impedance of an open tube of length $L_F$ and area $A_F$:

$$Z_F = j \frac{\rho_0 c}{A_F} \tan k L_F$$

Natural frequencies are those values of $k = \omega/c = 2\pi f/c$ for which

$$\frac{1}{A_B} \cot k L_B = \frac{1}{A_F} \tan k L_F$$

or

$$\cot k L_B = \frac{A_F}{A_B} \tan k L_F$$

(34)
Noting that $\tan x = \sin x / \cos x$ and that $\cot y = \cos y / \sin y$, Eq. 34 can be written as

$$\cos kL_B \cos kL_F = \frac{A_B}{A_F} \sin kL_B \sin kL_F$$

This equation implies that if $k$ satisfies Eq. 34 when $L_B = X$ and $L_F = Y$, $k$ also satisfies Eq. 34 when $L_B = Y$ and $L_F = X$.

The values of $k$ which satisfy this equation specify the “natural frequencies” ($f = kc/(2\pi)$) of the two tube system. Unfortunately, in general this equation cannot be solved analytically, so resort must be made to numerical approximation or graphical methods.

Approximate solutions to Eq. 34 may be obtained graphically by plotting $\cot kL_B$ and $A_B/A_F \tan kL_F$ as functions of $f$ and estimating points of intersection (Fig. 33). The five lowest natural frequencies for the cases $L_B = 5$, $L_F = 12$ cm, and $L_B = 12$, $L_F = 5$ cm, $A_F/A_B = 3.00$ and $A_B/A_F = 0.33$ are listed in Table 8.2. Note that the natural frequencies for the case $L_B = 5$, $L_F = 12$ cm are the same as the natural frequencies for the case $L_B = 12$, $L_F = 5$ cm, although the graphs appear to be different.

It can be shown that the natural frequencies of the two-tube system must satisfy

$$\cos kL = \frac{A_B - A_F}{A_B + A_F} \cos k (L_B - L_F)$$

which, at the point $L_B = L_F = L/2$ implies that the natural frequencies are given by

$$f = \frac{c}{2\pi L} \arccos \frac{A_B - A_F}{A_B + A_F}.$$ 

Solutions for the natural frequencies in the cases $A_B/A_F = 3.0$ and $A_B/A_F = 0.33$ are shown in Fig. 34.

The dependence of these natural frequencies on $A_F/A_B$ when $L_B = 5$, $L_F = 12$ cm is shown in Fig. 35. Also shown in this figure are the limiting values of these frequencies in the case $A_F/A_B \to 0$ and $A_F/A_B \to \infty$.

The dependence of these natural frequencies on $L_B$ for a $L_B + L_F = 17$ cm long tube configuration is shown in Fig. 36 in the case $A_B/A_F = 0.33$ and in Fig. 37 in the case $A_B/A_F = 3.00$.

It is clear that the natural frequencies of the a tube such as the vocal tract depends both on the location of the region of narrowing and the degree of narrowing.

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6Using the trigonometric identity $\cos (x - y) = \cos (x) \cos (y) + \sin (x) \sin (y)$. 

37
Figure 33: Illustrating the solution to Eq. 34 Solid curves represent the left side of the equation, with asymptotes are indicated by dotted vertical lines. Dashed curves represent the right side of the equation, with wider dashed vertical lines indicating the asymptotes. Filled squares on the Frequency axis indicate the natural frequencies of the two-tube system. The upper panels display results for \( L_B = 5 \text{ cm}, L_F = 12 \text{ cm} \). The lower panels display results for \( L_B = 12 \text{ cm}, L_F = 5 \text{ cm} \). The left panels display results for \( A_B/A_F = 3.0 \); The right panels display results for \( A_B/A_F = 0.33 \);
Table 1: Estimated natural frequencies for the systems analyzed in Fig. 33. Lengths are given in cm, frequencies in Hz.

<table>
<thead>
<tr>
<th>$L_B$</th>
<th>$L_F$</th>
<th>$A_B/A_F$</th>
<th>$F_1$</th>
<th>$F_2$</th>
<th>$F_3$</th>
<th>$F_4$</th>
<th>$F_5$</th>
</tr>
</thead>
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<tr>
<td>5</td>
<td>12</td>
<td>3.00</td>
<td>350</td>
<td>1460</td>
<td>2680</td>
<td>3490</td>
<td>4410</td>
</tr>
<tr>
<td>5</td>
<td>12</td>
<td>0.33</td>
<td>620</td>
<td>1590</td>
<td>2360</td>
<td>3555</td>
<td>4700</td>
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<tr>
<td>12</td>
<td>5</td>
<td>3.00</td>
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<td>1460</td>
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<td>12</td>
<td>5</td>
<td>0.33</td>
<td>620</td>
<td>1590</td>
<td>2360</td>
<td>3555</td>
<td>4700</td>
</tr>
</tbody>
</table>

Figure 34: Solution for the natural frequencies for an $L = 17$ cm two tube system in the case of $A_B/A_F = 3.0$ (abscissae of points of intersection of the dotted line with cosinusoidal curve) and $A_B/A_F = 0.33$ (abscissae of points of intersection of the dashed line with cosinusoidal curve) when $L_B = L_F = L/2 = 8.5$ cm.
Figure 35: Illustrating the dependence of the solution to Eq. 34 on $A_F/A_B$ in the case $L_B = 5$ cm, $L_F = 12$ cm. The solid circles on the right correspond to the limiting cases described by Eqs. 29 and 30, i.e. $A_B/A_F \to 0$. The solid circles on the left correspond to the limiting cases described by Eqs. 31 and 32, i.e. $A_F/A_B \to 0$. The dotted curve corresponds to the Helmholtz natural frequency (Eq. 33).
Figure 36: Illustrating the dependence of the solution to Eq. 34 on $L_B$ and $L_F$ in the case $A_F/A_B = 0.33$. 

$L_B$ $[L_F + L_B = 17 \text{ cm}]$
Figure 37: Illustrating the dependence of the solution to Eq. 34 on $L_B$ and $L_F$ in the case $A_B/A_F = 3.00$. 

$L_B \quad [L_F + L_B = 17 \text{ cm}]$
Duplicate natural Frequencies

When the coupling between the two tube segments is small enough, the natural frequencies of the coupled system of tubes may be approximated by finding the natural frequencies of the individual components. When the coupling is not zero, i.e. when $A_B/A_F \neq 0$, the coupling between the back and front sections of the tube must be taken into account.

When $0 < A_B \ll A_F < +\infty$, the natural frequencies of the two tube system are shifted slightly from the frequencies of the individual tubes (see Fig. 38). The greatest shift occurs when the natural frequencies of the front and back tube sections are equal. Under these conditions, the natural frequencies of the combined tube system are not equal but are split apart, one being higher than the natural frequency of each tube individually, the other lower.

If the natural frequencies of the uncoupled tubes is $f_0$, then in the case where $A_B/A_F << 1$ the natural frequencies of the combined system are approximately

$$f'_0 = f_0 \left( 1 \pm \frac{2}{\pi} \sqrt{\frac{A_B}{A_F}} \right)$$
9 Internal Sources.

Many speech sounds are generated by sources within the vocal tract rather than solely at the larynx. We show that for sources located with an acoustic tube, the transfer function between the source and the open end of the tube must necessarily have zeroes as well as poles.

Consider a sound velocity source located in the interior of a system of tubes (Fig. 39).

\[
\frac{U_F}{U_S} = \frac{Z_B}{Z_B + Z_F}
\]

\[
\frac{U_O}{U_F} = S_{21}
\]

so

\[
\frac{U_O}{U_S} = \frac{U_O U_F}{U_F U_S} = S_{21} \frac{Z_B}{Z_B + Z_F}
\]

At any frequency for which \(Z_B = 0\), the transfer function \(U_O/U_S\) will also be zero.

9.1 Source in a Uniform Tube

Consider a tube of length \(L\) that is closed at one end and open at the other with a volume velocity source \(U_S\) located at a distance \(L_B\) from the closed end (and \(L_F = L - L_B\) from the open end).
Then

\[ Z_B = -jz_0 \cot kL_B \]
\[ Z_F = jz_0 \tan kL_F \]
\[ S_{21} = \frac{1}{\cos kL_F} \]

so that

\[ \frac{U_O}{U_S} = \frac{1}{\cos kL_F} \times \frac{\cot kL_B}{\cot kL_B - \tan kL_F} \]
\[ \frac{U_O}{U_S} = \frac{\cos kL_B}{\cos kL} \]

Some observations:

1) Note that the transfer function \( U_O/U_S \) for a source located within a uniform tube reduces to the transfer function for a source located at the closed end of the tube in the case where \( kL_B \to 0 \).

2) Note that this transfer function differs from that which would apply if the source were at the closed end of the tube by the \( \cos kL_B \) in the numerator. This term introduces zeroes to the transfer function at frequencies

\[ f_Z = \frac{2n - 1}{4} \frac{c}{L_B} \text{ for } n = 1, 2, \ldots \]

3) The form of the transfer function isn’t really surprising. The poles of the transfer function are the natural frequencies of the tube system (closed at one end and open at the other). The zeroes of \( U_O/U_S \) are the zeroes of the impedance \( Z_B \). As long as \( Z_F \) is non-zero, at a frequency for which \( Z_B = 0 \), the entire source volume velocity \( U_S \) flows into \( Z_B \) leaving none to flow through \( Z_F \). As a result, we can deduce that in general the zeroes of \( Z_B \) will be the zeroes of \( U_O/U_S \).
12  Perturbations

This section considers the effects of perturbations, small changes in the shape of an acoustic tube system, on the natural frequencies of the system. We begin by considering a modification to a simple acoustic circuit, then we consider the effect of adding an acoustic mass at the open end of a uniform acoustic tube, then we consider the effect of a small hole at the closed end of an acoustic tube, and finally we consider the general case.

12.1  Effects of Changes in Area on Acoustic Mass and Compliance

Consider a change in the cross sectional area of a uniform tube from \( A \) to \( A + \Delta A \). If the change occurs over a sufficiently short length of tube \( L \) and at a maximum in volume velocity \( U \) (and a minimum in pressure \( P \)), the change in the acoustic mass of the tube \( M (A) = \rho_0 L / A \) is more important than the change in compliance. But, for a small change in cross sectional area,

\[
M (A + \Delta A) = \frac{\rho_0 L}{(A + \Delta A)} \approx \frac{\rho_0 (L - \Delta L)}{A}
\]

provided

\[
\Delta L = L \frac{\Delta A}{A}.
\]

Thus an increase in area at a relative maximum of volume velocity has roughly the same effect as a shortening of the tube at constant area. For a uniform tube this implies that the natural frequencies of the tube will increase (slightly).

If, on the other hand, the change occurs over a sufficiently short length of tube \( L \) but at a maximum in pressure \( P \) (and a minimum in volume velocity \( U \)), the change in the acoustic compliance of the tube \( C (A) = AL / (\rho_0 c^2) \) is more important than the change in mass. But, for a small change in cross sectional area,

\[
C (A + \Delta A) = \frac{(A + \Delta A) L}{\rho_0 c^2} \approx \frac{A (L + \Delta L)}{\rho_0 c^2}
\]

provided again

\[
\Delta L = L \frac{\Delta A}{A}.
\]

Thus an increase in area at a relative maximum of pressure has roughly the same effect as a lengthening of the tube but keeping the area constant. For a uniform tube this implies that the natural frequencies of the tube will decrease (slightly).

12.2  Perturbation of an MC Circuit.

Consider the simple acoustic circuit in Fig. 47. We have seen that at its natural frequency

\[
f_0 = \frac{\omega_0}{2\pi} = \frac{1}{2\pi \sqrt{MC}}
\]
Figure 47: Circuit representation of an acoustic circuit with an acoustic mass of $M$ and an acoustic compliance of $C$.

Figure 48: Circuit representation of a perturbed acoustic circuit with a total acoustic mass of $M + \Delta M$.

It is possible for there to exist a volume velocity $u(t)$ and pressure $p_M(t)$ within the circuit provided they are sinusoidal and of frequency $f_0$

$$u(t) = U \sin 2\pi f_0 t$$
$$p_M(t) = P_M \cos 2\pi f_0 t$$

Because

$$p_M(t) = M \frac{d}{dt} u(t)$$

we must have

$$P_M = 2\pi f_0 M U.$$
In this case the average energy stored in the acoustic mass

\[ \langle E_M \rangle = \frac{1}{4} M U^2 \]

equals the average energy stored in the acoustic compliance

\[
\langle E_C \rangle = \frac{1}{4} C P_M^2 \\
= \frac{1}{4} C (2\pi f_0 M U)^2 \\
= \frac{1}{4} C \frac{4\pi^2}{(2\pi \sqrt{MC})^2} M^2 U^2 \\
= \frac{1}{4} M U^2 \\
= \langle E_M \rangle 
\]

If now the circuit is “perturbed” by increasing the acoustic mass by \( \Delta M \) we know that the natural frequency of the circuit will decrease to

\[ f'_0 = \frac{\omega'_0}{2\pi} = \frac{1}{2\pi} \frac{1}{\sqrt{(M + \Delta M) C}} \]

But the increase in acoustic mass tends to increase \( \langle E_M \rangle \).

\[ \langle E_M \rangle = \frac{1}{4} (M + \Delta M) U^2 \]

To still be at a natural frequency \( \langle E_C \rangle \) must increase (e.g., Sec. 9.2 of \textit{Notes on Acoustic Circuits}).

\[ \langle E_C \rangle = \frac{1}{4} C P_C^2 \]

Increasing \( P_C \) at constant \( U \) requires that the magnitude of the impedance of the compliance, \( 1/(2\pi f C) \), must increase, which in turn means that the natural frequency must decrease, so

\[ f'_0 < f_0 \]

We will see in the general case that arguments based on stored energy allow the effects of perturbations to be determined in general.

\section*{12.3 Radiation}

Some insight can be gained by considering the effect of radiation at the mouth on the natural frequencies of the vocal tract. We showed previously that accounting for the radiation mass tended to lower all natural frequencies. That this should be the case can be seen by
recognizing that in the unmodified a tube, at each of the natural frequencies, the volume velocity is a maximum at the open end of a tube. If this volume velocity moves an acoustic mass at the end of the tube, there is an effective positive reactance that must be balanced by a negative reactance for the tube to be resonant at its (new) natural frequencies. This can only be achieved if the new natural frequencies are lower than the original natural frequencies.

12.4 Small Hole in the Closed End

Consider a small hole of area $A_H$ and length $L_H$ in the closed end of a tube of cross-sectional area $A_T$. The hole changes the infinite acoustic impedance originally at the end to that of a finite acoustic mass, $M_H$

$$M_H = \frac{\rho_0 L_H}{A_H}$$

At a distance $\Delta L$ from the hole the air in the tube acts as a compliance in parallel with the acoustic mass (the pressure in the tube can either compress the air in the tube or cause air to flow through the hole):

$$C_A = \Delta L \frac{A_T}{\rho_0 c^2}$$

The impedance becomes infinite when:

$$0 = \frac{1}{j\omega C_A} + j\omega M_H$$

$$= \frac{1}{j\omega \frac{A_T}{\rho_0 c^2} \Delta L} + j\omega \frac{L_H}{A_H}$$

or when

$$\Delta L = \left(\frac{c}{\omega}\right)^2 \frac{A_H}{A_T} \frac{1}{L_H}$$

Thus the effect of the hole is to shorten the tube by $\Delta L$ and to raise the natural frequencies of the tube.

12.5 General Case

A theorem that generalizes the $MC$ result in Sec. 12.2 states that, in a acoustic circuit containing masses and compliances, a perturbation in the mass or compliance values that causes an increase in the average stored energy at a given natural frequency results in a decrease in that natural frequency. For small changes, the shift in the natural frequency is proportional to the change in average stored energy (Staelin et al., 1994).

Although this theorem applies to a wide variety of acoustic systems, we restrict our attention to the system consisting of a uniform tube closed at one end (Fig. 49). We assume that at a distance $L_C$ from the closed end, the area of the tube is increased or decreased for
Figure 49: Illustrating a perturbation in a uniform tube that is closed at one end and opened at the other. The tube is of length $L$ and cross-sectional area $A$. The perturbation in the tube consists of a constriction which reduces the area of the tube to $A_C$ at a distance of $L_C$ from the closed end of the tube.

A short distance ($\Delta l$). We also assume that for this small perturbation (from $A$ to $A_C$) the distributions of pressure and velocity amplitude at the natural frequencies remain unchanged.

The shift in natural frequency due to a localized perturbation in the cross-sectional area of the vocal tract depends on the distribution of sound pressure and volume velocity amplitude along the tract (Fig. 50).

The average stored energy associated with volume velocity $U(L_C)$ in a short length of tube ($\Delta l$) at a distance $L_C$ from the closed end of the tube is

$$\langle E_U \rangle = \frac{1}{4} |U(L_C)|^2 \frac{\rho_0 \Delta l}{A}$$

so

$$\frac{d \langle E_U \rangle}{dA} = -\frac{1}{4} |U(x)|^2 \frac{\rho_0 \Delta l}{A^2} = -\frac{1}{A} \langle E_U \rangle$$

This indicates that the change of the energy associated with volume velocity stored in a short length of tube caused by a change in the area $\Delta A$ is proportional to the the energy associated with pressure, the proportionality factor being the negative of fractional change in area $\Delta A/A$. Similarly, it can be shown that the change of the energy associated with pressure stored in a short length of tube caused by a change in the area $A$ is proportional to the the energy associated with pressure with the proportionality being the fractional change in area.

At a point $L_C$ where the magnitude of the volume velocity is close to a maximum (and the sound pressure is near zero), a reduction in cross-sectional area causes an decrease in the
Figure 50: The solid curve represents the amplitude of pressure ($|P(x)|$) and dotted curve represents the amplitude of volume velocity ($|U(x)|$) within a tube closed at one end and open at the other.

natural frequency. This can be seen because

1. The derivative of the energy associated with pressure is near zero.

2. The derivative of the energy associated with volume velocity is negative, as is the fractional change in area.

If the cross-sectional area is reduced at a point where the sound pressure is a maximum and the volume velocity near zero, the natural frequency increases.

At an intermediate location, where the sound pressure amplitude and the volume velocity amplitude are between their maximum and minimum values, a perturbation in the cross-sectional area causes no change in the natural frequency.

The changes in the first three natural frequencies of the uniform tube is shown in Fig. 51. At a particular point for any one of these functions, the ordinate is proportional to the change in the natural frequency that results from a decrease in cross-sectional area over a small length located at this point.

This general rule applies not only to the uniform tube but to an arbitrary configuration that is open at one end, since there is a minimum in sound pressure and a maximum in volume velocity at this point.

The curves show that a decrease in cross-sectional area at the open end, corresponding to the lip opening, causes a downward shift in all formants. This change in area approximates the change in vocal tract configuration for a vowel when the lip opening is reduced, as for rounding.
Figure 51: Curves showing the relative direction and magnitude of the change in the lowest three natural frequencies ($F_1$, $F_2$, and $F_3$) for a uniform tube when the cross-sectional area is reduced at a point along the tube. A $-$ indicates that the natural frequency is reduced, a $+$ indicates that the natural frequency is increased.

Likewise, at the closed end, near the glottis, a decrease in cross-sectional area causes a rise in all the formant frequencies. In the case of the uniform tube, all curves have a zero at the midpoint, indicating that a perturbation in cross-sectional area at this point has no effect on any natural frequency.

A reduction in area at any point in the anterior (front) half of the uniform tube produces a drop in $F_1$, the effect being greater when the perturbation is closer to the open end. Similarly a reduction in area in the posterior (back) half of the uniform tube results in an increase in $F_1$. 
15 Vowel Production

The natural frequencies of the vocal tract are usually referred to as *formants* and designated by \( F_1 < F_2 < F_3 \) etc. By common usage the frequency of the pulses of air emitted by the glottis is called the *fundamental frequency* and is designated by \( F_0 \).

In a neutral vowel, the vocal tract is well represented by a uniform acoustic tube that is closed at the glottal end at open at the lip end and for which

\[
F_n = (2n - 1) \frac{c}{4L} \quad \text{for } n = 1, 2, \ldots
\]

where \( L \) is the length of the vocal tract (typically 17 cm for adult men, shorter for women and children) and \( c \) is the speed of sound (typically 34000 cm/sec) so that \( F_n = 500, 1500, 2500, \ldots \) Hz.

There is some evidence (e.g., Carlson et al., 1970) that vowels can be identified on the basis of the values of the first two formants. Consequently we will confine our discussion to the mechanisms that are responsible for establishing the values of \( F_1 \) and \( F_2 \). Those interested in a more detailed discussion are referred to Stevens (1999).

15.1 \( F_1 \) - First Formant Frequency

The frequency of the first formant \( F_1 \) is determined by the degree to which the tongue is raised from its neutral position within the vocal tract (corresponding to the frequency of the
neutral vowel $c/(4L)$.

Figure 57: Vocal tract shape used by an adult male speaker to produce the high vowels /i/ (left) and /u/ (right). Adapted from Perkell (1969).

**High Vowels**

The vowels /i/ and /u/ are called "high" vowels because they are produced with the tongue raised toward the roof of the mouth (Fig. 57). The vowels are typically produced with a narrowing of the cross-sectional area in the front part of the vocal tract and a widening of the cross-sectional area toward the glottis.

The combination of the acoustic compliance of the region behind the narrowing (typically 50 cc for males) and the acoustic mass of the narrow air passage (typically of 4 cm length and 0.4 sq cm area) gives rise to a low-frequency first formant, _a la_ a Helmholtz resonator.

The spectra for /i/ and /u/ (Fig. 58 and 59) are typically much narrower and more shallow than the spectra of non-high vowels (e.g., /a/) with low-frequency dips in the spectrum below the first formant, that is $F_1$ is only 100-200 above $F_0$.

**Non-High Vowels**

For non-high vowels, $F_1$ is greater than for high vowels, as is the spacing between $F_0$ and $F_1$, e.g. Figs. 58 and 59. Because of this, the spectrum of these vowels (e.g. the right panel of Fig. 58) exhibits a clear peak in the region of $F_1$.

There are two classes of non-high vowels: low vowels (e.g., /ae/ and /a/) which have a relatively high $F_1$ and non-low vowels, e.g., /e/ and /o/, with $F_1$ intermediate between high and low vowels.

**Low Vowels**

Low vowels are typically produced with a narrowing of the back half of the vocal tract (e.g., Fig. 60). As a result $F_1$ is higher than for a neutral vowel. Depending on whether the back
Figure 58: Spectra of the synthetic vowels /i/ (left panels), /u/ (center panels), and /ah/ (right panels) with formant frequencies appropriate for an adult female (upper panels) and male (lower panels) talkers. The time waveforms and time windows used in the spectral analysis are also shown. (Adapted from Klatt and Klatt, 1990).

part of the vocal tract is longer or shorter than the front part $F_1$ is primarily associated with the back or front portions of the vocal tract. When the lengths are about equal, $F_1$ is higher when the ratio $A_B/A_F$ decreases.

**Intermediate Vowels**

For intermediate vowels such as /e/ and /o/ the tongue body is at an intermediate height relative to high and low vowels (Fig. 61), so that the cross-sectional area of the region of narrowing is increased. This causes a decrease in the acoustic mass of the narrowing and a consequent increase in $F_1$. The value of $F_1$ is thus generally between that for high and low vowels.
Figure 59: Spectrograms of 100 ms portions of the three vowels in Fig. 58. (From Klatt and Klatt, 1990.)

Figure 60: Vocal tract shapes used by an adult male speaker of English to produce the low vowels /a/ (left) and /ae/ (right). (Adapted from Perkell, 1969.)

Figure 61: Vocal tract shapes used by an adult male speaker of French to produce the high vowels /e/ (left) and /o/ (right). (Adapted from Bothorel et al., 1986.)
15.2 $F_2$ - Second Formant Frequency

The front-back position of the tongue body largely determines the value of $F_2$, the second formant frequency of the vowel. The pairs of vocal tract shapes in Figs. 57, 60, and 61, show how vowels can have different front-back positions yet have roughly the same tongue height. The acoustic consequences of different front-back shapes is roughly the same for different tongue body heights.

![Figure 62: Left: vocal tract shapes used by an adult male speaker to produce the low vowels /a/ and /ae/. Right: Two tube model for the production of low vowels. (Adapted from Perkell, 1971, and Stevens, 1999.)](image)

Low Vowels

Although low vowels are typically produced with a relatively narrow back cavity and a relatively wide front cavity, the position of the narrowing (i.e. $L_B/L_F$) can be adjusted somewhat to produce the different vowels /ae/ and /a/ (Fig. 62). The frequencies of $F_1 \cdots F_4$ are plotted as a function of $L_B$ in Fig. 63. While $F_1$ is relatively fixed, $F_2$ changes significantly with $L_B$.

High Vowels

As in the case of low vowels, there is a fronted tongue body position for which $F_2$ has a relatively high value that is close to $F_3$, and a backed tongue body position which, under certain conditions, causes $F_2$ to have a relatively low value that is close to $F_1$.

Representative shapes of the vocal tract are shown in Fig. 57: the high vowels are characterized by a narrowing in the front of the vocal tract and a widening in the back. The extent of the narrow region is variable, however, and can be modelled as the three-tube configuration of Fig. 64. Detailed calculations of the dependence of $F_1 \cdots F_4$ on $L_B$ with $L_C = 5$ cm are shown in Fig. 65. It can be seen that $F_1$ is in the range 200-300 Hz, $F_2$ has a broad maximum at about $L_B = 8$ cm,
Figure 63: First four natural frequencies for the two-tube configuration of Fig. 62 as a function of the length ($L_B$) of the back section. The total length $L = L_B + L_F = 16$ cm and $A_F = 3$ sq cm. Dashed curves correspond to $A_B = 0$, solid curves to $A_B = 0.5$ sq cm. (Adapted from Stevens, 1989.)

Figure 64: Three-tube model of the vocal tract shape used by an adult male speaker to produce the high vowels /i/ and /u/. (Adapted from Stevens, 1989.)

Intermediate Vowels

The dependence of $F_2$ on the locus of the narrowing formed by the tongue relative to the palate is similar to that seen for the high vowels, (Fig. 65). Because the narrowing is less extreme than for the high vowels, there is more coupling between the narrow and wide portions of the vocal tract and consequently a greater separation between $F_2$ and $F_3$.

Summary

Movement of the tongue body toward the lips tends to increase $F_2$ to a maximum value for a given tongue height. The maximum is greater for high vowels than for low vowels. Front vowels are characterized by a broad minimum in the spectrum (an absence of formants) in the mid frequencies (between the first and second natural frequencies). Movement of the tongue body toward the larynx, causes $F_2$ to be displaced towards a relatively low value that
Figure 65: The middle panel: the natural frequencies of the three-tube configuration of Fig. 64 as a function of $L_B$, with $L_C = 5$ and $L_B + L_C + L_F = 16$ cm. The cross-sectional areas assumed in the computation were $A_1 = 3$, $A_C = 0.3$, and $A_F = 1.0$ sq cm. The portions of the curves corresponding to non-low front vowels ($L_B = 6$ and 11 cm) are marked by vertical lines. Spectra corresponding to $L_B = 8$ and 10 cm are shown. (Adapted from Stevens, 1999.)

is close to $F_1$. A consequence of this is that the amplitudes of $F_3$ and above are low relative to the amplitudes of $F_1$ and $F_2$.

15.3 Perturbation Theory

It is instructive to determine whether the changes in formants are consistent with the predictions of perturbation theory.

Relative to a uniform tube, for which the formant frequencies are:

$$F_n = \frac{(2n - 1)c}{4L} \quad \text{for } n = 1, 2, \ldots$$

perturbation theory (Fig. 66) provides an account of how small changes in the same of the tube affect the formant frequencies.

1. Narrowing the cross-sectional area in the front of the tube, or widening it in the back, cause $F_1$ to be reduced from $c/(4L)$.

2. If the narrowing occurs towards the middle of the front half of the tube, $F_2$ will increase relative to $3c/(4L)$. If the narrowing occurs towards the middle of the back half of the tube, $F_2$ will decrease relative to $3c/(4L)$.

3. If the narrowing occurs at the very front of the tube, $F_2$ will decrease relative to $3c/(4L)$.

These shifts are roughly consistent with the shifts in formants seen in Fig. 56.
Figure 66: Curves showing the relative direction and magnitude of the change in the lowest three natural frequencies ($F_1$, $F_2$, and $F_3$) for a uniform tube when the cross-sectional area is reduced at a point along the tube. A $-$ indicates that the natural frequency is reduced, a $+$ indicates that the natural frequency is increased.

- Narrowing the uniform tube in the front of the tube, or widening the tube in the back, shifts $F_1$ from its neutral value $c/(4L)$ Hz toward the left of Fig. 56, consistent with /i/ and /u/.

- If this narrowing is made near the middle of the front half of the tube, $F_2$ will be increased to a greater value than its neutral value $3c/(4L)$, towards /i/ in Fig. 56. If this narrowing is made near the front of the tube, $F_2$ will be decreased to a lower value than $3c/(4L)$ Hz, towards /u/ in Fig. 56.

- Narrowing the uniform tube in the back of the tube, or widening the tube in the front, shifts $F_1$ from $c/(4L)$ Hz toward the right of Fig. 56, consistent with /a/ and /ae/.

- If this narrowing is made near the middle of the back half of the tube, $F_2$ will be decreased to a lower value than $3c/(4L)$, towards /a/ in the diagram. If this narrowing is made near the back of the tube, and accompanied by a widening toward the front, $F_2$ will be increased to a higher value than $3c/(4L)$ Hz, towards /ae/ in Fig. 56.

Thus, perturbation theory provides a rough guide to the formant structure of the vowels /i/, /u/, /a/, and /ae/.
References


