Slides for Acoustic Tubes and Vowels.*

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October 29, 2012
Figure 1: Schematic representation of the outer, middle, and inner ear.
Figure 2: Schematic representation of the vocal tract.
Figure 3: A lumped element approach to modelling a distributed system, such as an acoustic tube of uniform cross section

\[ M \Delta x = \frac{\rho_0}{A} \Delta x \]

\[ C \Delta x = \frac{A}{B_A} \Delta x = \frac{A}{\gamma P_0} \Delta x \]
\[ u(x - \Delta x, t) - u(x, t) = C\Delta x \frac{\partial p(x, t)}{\partial t} \]
\[ p(x, t) - p(x + \Delta x, t) = M\Delta x \frac{\partial u(x, t)}{\partial t} \]
\[ -\frac{\partial u(x, t)}{\partial x} = C\frac{\partial p(x, t)}{\partial t} \]
\[ -\frac{\partial p(x, t)}{\partial x} = M\frac{\partial u(x, t)}{\partial t} \]
Figure 4: A distributed element approach to modelling sound propagation in a small segment of the cochlea. The fluid masses are represented by $N$ and the mechanical properties of the basilar membrane by $M$, $C$, and $R$ elements. All element values are functions of position along the basilar membrane $x$. 
\[
\frac{\partial^2 p(x, t)}{\partial x^2} = MC \frac{\partial^2 p(x, t)}{\partial t^2}
\]

\[
\frac{\partial^2 u(x, t)}{\partial x^2} = MC \frac{\partial^2 u(x, t)}{\partial t^2}
\]

\[
p(x, t) = p_+(x - ct) + p_-(x + ct)
\]

\[
u(x, t) = u_+(x - ct) - u_-(x + ct)
\]

\[
u(x, t) = \frac{1}{z_0} p_+(x - ct) - \frac{1}{z_0} p_-(x + ct)
\]

\[
MC = \frac{\rho_0}{\gamma P_0} = \frac{1}{c^2}
\]

\[
\sqrt{\frac{M}{C}} = \frac{\rho_0 c}{A} = z_0
\]

\[
c \times z_0 = \frac{1}{C}
\]

\[
\frac{c}{z_0} = \frac{1}{M}
\]
\[ p(t) = P(x)e^{j\omega t} \]
\[ u(t) = U(x)e^{j\omega t} \]

\[ \frac{dP(x)}{dx} = -j\omega MU(x) \]
\[ \frac{dU(x)}{dx} = -j\omega CP(x) \]

\[ \frac{d^2 P(x)}{dx^2} = -\omega^2 MCP(x) \]
\[ \frac{d^2 U(x)}{dx^2} = -\omega^2 MCU(x) \]
\[ k = \omega \sqrt{MC} = \frac{\omega}{c} \]

\[ \frac{d^2 U(x)}{dx^2} = -\omega^2 MC U(x) = -\left(\frac{\omega}{c}\right)^2 U(x) = -k^2 U(x) \]

\[ U(x) = U_0 \sin (kx + \theta), \]

\[ P(x) = \frac{1}{-j\omega C} \frac{dU(x)}{dx} = jz_0 U_0 \cos (kx + \theta), \]
\[ k = \omega \sqrt{MC} = \frac{\omega}{c} \]

\[ \frac{d^2 P(x)}{dx^2} = -\omega^2 MC P(x) = -\left(\frac{\omega}{c}\right)^2 P(x) = -k^2 P(x) \]

\[ P(x) = P_0 \sin (kx + \phi), \]

\[ U(x) = \frac{1}{-j\omega M} \frac{dU(P)}{dx} \]
\[ = j \frac{P_0}{z_0} \cos (kx + \phi), \]
\[ k = \omega \sqrt{MC} = \frac{\omega}{c} \]

\[ \frac{d^2U(x)}{dx^2} = -\omega^2 MC U(x) = -\left(\frac{\omega}{c}\right)^2 U(x) = -k^2 U(x) \]

\[ P(x) = \frac{1}{-j\omega C} \frac{dU(x)}{dx} \]

\[ U(x) = U_0 \sin (kx) \quad U(x) = U_0 \cos (kx) \]

\[ P(x) = j\varepsilon_0 U_0 \cos (kx) \quad P(x) = -j\varepsilon_0 U_0 \sin (kx) \]
\[ k = \omega \sqrt{MC} = \frac{\omega}{c} \]

\[ \frac{d^2 P(x)}{dx^2} = -\omega^2 MC P(x) = -\left(\frac{\omega}{c}\right)^2 P(x) = -k^2 P(x) \]

\[ U(x) = \frac{1}{-j\omega M} \frac{dU(P)}{dx} \]

\[ P(x) = P_0 \sin (kx) \quad P'(x) = P_0 \cos (kx) \]

\[ \underline{U}(x) = j \frac{P_0}{z_0} \cos (kx) \quad \underline{U}(x) = -j \frac{P_0}{z_0} \sin (kx) \]
Case 1: Tube open at both ends.

\[ P(-L) = 0 \quad \text{and} \quad P(0) = 0 \]

Figure 5: A tube of length \( L \) open to the atmosphere at both ends.

First boundary condition: \( \sin \phi = 0 \) or \( \phi = 0 \).
Second boundary condition: \( \sin (-kL) = -\sin kL = 0 \).

\[ k_n = n\frac{\pi}{L} \quad \text{where} \quad n = 0, 1, 2, 3, \ldots \]
\[ f_n = n\frac{c}{2L} \quad \text{where} \quad n = 0, 1, 2, 3, \ldots \]
Figure 6: The solid curve represents the amplitude of pressure ($|P(x)|$) and dotted curve represents the amplitude of volume velocity ($|U(x)|$) within a tube open at both ends. The curves are drawn under the assumption that $z_0 = 1$ acoustic ohm.

\[
P(x) = P_0 \sin \left( n\pi \frac{x}{L} \right)
\]

\[
U(x) = j \frac{P_0}{z_0} \cos \left( n\pi \frac{x}{L} \right)
\]
Case 2: Tube closed at both ends.

\[ U(-L) = 0 \quad U(0) = 0 \]

\[ x = -L \quad x = 0 \]

Figure 7: A tube closed at both ends.

\[ U(0) = 0 \]
\[ U(-L) = 0 \]

First boundary condition: \( \sin \theta = 0 \) or \( \theta = 0 \),

Second boundary condition: \( \sin -kL = -\sin kL = 0 \).

\[ k_n = n \frac{\pi}{L} \quad \text{where } n = 0, 1, 2, 3, \ldots \]
\[ f_n = n \frac{c}{2L} \quad \text{where } n = 0, 1, 2, 3, \ldots \]
Figure 8: The solid curve represents the amplitude of pressure (|P(x)|) and dotted curve represents the amplitude of volume velocity (|U(x)|) within a tube closed at both ends. The curves are drawn under the assumption that z₀ = 1 acoustic ohm.

\[
U(x) = U_0 \sin \left( n\pi \frac{x}{L} \right) \\
P(x) = jz_0 U_0 \cos \left( n\pi \frac{x}{L} \right)
\]
Case 3: Tube closed at one end and open at the other.

\[ U(-L) = 0 \quad \text{and} \quad P(0) = 0 \]

First boundary condition: \( \sin \theta = 0 \) so \( \theta = 0 \).

Second boundary condition:

\[
0 = U(-L) = \left. \frac{1}{-j\omega M} \frac{dP}{dx} \right|_{x=-L} = j \frac{P_0}{Mc} \cos kL
\]

\[
\cos kL = 0
\]

\[
k_n = (2n + 1) \frac{\pi}{2L}, \quad \text{where} \quad n = 0, 1, 2, \ldots,
\]

\[
f_n = (2n + 1) \frac{c}{4L}, \quad \text{where} \quad n = 0, 1, 2, \ldots
\]
Figure 10: The solid curve represents the amplitude of pressure ($|P(x)|$) and dotted curve represents the amplitude of volume velocity ($|U(x)|$) within a tube closed at one end and open at the other.

\[
U(x) = U_0 \cos \left( \frac{2n - 1}{2L} \pi x \right)
\]

\[
P(x) = -j \rho_0 U_0 \sin \left( \frac{2n - 1}{2L} \pi x \right)
\]
1 System Functions for Tubes

1.1 Tube Open at One End.

\[ U(-L) = U_s \]

First boundary condition \( P(0) = 0 \)

\[
\sin \theta = 0 \\
\theta = 0
\]

Second boundary condition: \( U(-L) = U_s \)

\[
U_0 \cos kL = U_s
\]

Figure 11: A tube open at one end (\( x = 0 \)) and driven by a volume velocity source at the other (\( x = -L \)).
Figure 12: The solid curve represents the dependence of the normalized reactance $X' = X/z_0$ on normalized frequency $\Omega = \omega/\omega_0$, where $\omega_0 = c/L$, for a tube open at one end and driven by a volume velocity source at the other. The dashed vertical lines mark the frequencies of the (imaginary-valued) poles of $Z(-L)$.

$$Z(-L) = \frac{P(-L)}{U_S} = jz_0 \tan kL.$$
Figure 13: The solid curve represents the dependence of the magnitude of the volume velocity transfer ratio $|H|$ on normalized frequency $\Omega$ where $\Omega = \omega/\omega_0$ and $\omega_0 = c/L$ for a tube open at one end and driven by a volume velocity source at the other. The dashed vertical lines mark the frequencies of the (purely imaginary) poles of $H$.

$$H(j\omega) = \frac{U(0)}{U_S} = \frac{U_0}{U_0 \cos kL} = \frac{1}{\cos kL} = \frac{1}{\cos \omega_c} = \frac{1}{\cos \Omega}$$
Figure 14: Distribution of volume velocity (dashed curves) and pressure (solid curves) within a tube open at the right end and driven by a volume velocity source at the left end. The five panels correspond to the normalized frequencies $\Omega = 0.25\pi, 0.425\pi, 0.575\pi, 0.925\pi, 1.075\pi$, increasing from the bottom to the top. In each panel, the lower straight line corresponds to zero pressure/volume velocity. The volume velocity at the left end of the tube ($|U_S|$) is the same in all panels. The curves for pressure correspond to $z_0 = 1$ acoustic ohm. Note: the relative amplitudes of pressure and volume velocity are to scale.
1.2 Tube Closed at One End.

First boundary condition: $U(-L) = U_S$

Second boundary condition: $U(0) = 0$.

$$U(x) = U_0 \sin (kx + \theta)$$

$$P(x) = j\bar{z}_0 U_0 \cos (kx + \theta)$$

$\sin \theta = 0$

$\theta = 0$

$$-U_0 \sin kL = U_S$$
Figure 16: The solid curve represents the dependence of the normalized reactance $X/z_0$ on normalized frequency $\Omega$ where $\Omega = \omega/\omega_0$, where $\omega_0 = c/L$. for a tube closed at one end and driven by a volume velocity source at the other. The dashed vertical lines mark the frequencies of the (imaginary-valued) poles of $Z(-L)$.

$$Z(-L) = \frac{P(-L)}{U_S} = -jz_0 \cot kL$$
Figure 17: The solid curve represents the dependence of the magnitude of the normalized transfer impedance $|Z_T/z_0|$ on normalized frequency $\Omega$ where $\Omega = \omega/(c/L)$. for a tube closed at one end and driven by a volume velocity source at the other.

$$Z_T(j\omega) = \frac{P(0)}{U_S} = -jz_0 \frac{1}{\sin kL} = -jz_0 \frac{1}{\sin \Omega}$$
Figure 18: Distribution of volume velocity (dashed curves) and pressure (solid curves) within a tube closed at the right end and driven by a volume velocity source at the left end. The five panels correspond to the normalized frequencies $\Omega = 0.25\pi, 0.425\pi, 0.575\pi, 0.925\pi, 1.075\pi$, increasing from the bottom to the top. In each panel, the lower straight line corresponds to zero pressure/volume velocity. The volume velocity at the left end of the tube ($|U_S|$) is the same in all panels. The curves for pressure correspond to the value $z_0 = 1$ acoustic ohm. Note: the relative amplitudes of pressure and volume velocity are to scale.
$P_1 = Z_{11}U_1 + Z_{12}U_2$

$P_2 = Z_{21}U_1 + Z_{22}U_2$

$Z_{11} = \frac{P_1}{U_1} \bigg|_{U_2=0} = -jz_0 \cot kL$

$Z_{22} = \frac{P_2}{U_2} \bigg|_{U_1=0} = Z_{11} = -jz_0 \cot kL$

$Z_{21} = \frac{P_2}{U_1} \bigg|_{U_2=0} = Z_T = \frac{-jz_0}{\sin kL}$

$Z_{12} = \frac{P_1}{U_2} \bigg|_{U_1=0} = Z_{21} = Z_T = \frac{-jz_0}{\sin kL}$

Figure 19: A tube that is connected to two independent volume velocity sources, $U_1$ and $U_2$. 
Figure 20: Circuit model for the two-port described by Eq. 1 and 1.
Figure 21: A tube that is closed at one end, and connected to a lumped acoustic mass $M_R$ at the other.

Figure 22: Circuit model for the acoustic system in Fig. 21.
Figure 23: Circuit model for the acoustic system in Fig. 21 when $U_S = 0$.

\[ Z_R + Z_{12} + (Z_{22} - Z_{12}) = Z_R + Z_{22} = 0 \]

\[-jz_0 \cot kL + j\omega M_R = 0 \]

\[ \cot \left( \frac{\omega}{c/L} \right) = \frac{\omega}{c/L} \frac{cM_R}{z_0 L} \]

\[ \cot \Omega = \frac{M_R}{LM} \Omega = \mu \Omega, \]

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Figure 24: The solid curve represents the dependence of $\cot \Omega$, the negative of the normalized reactance $X/z_0$ of the tube on normalized frequency $\Omega = \omega/(c/L)$. The dashed straight line represents the reactance of the lumped element acoustic mass, and corresponds to the case $\mu = 1$ or $M_R = ML$. The frequencies of the filled circles are the natural frequencies of the acoustic system.
Figure 25: Prototypical Helmholtz Resonator.
Figure 26: The solid curve represents the dependence of the normalized reactance $X/z_0$ of an acoustic tube with acoustic mass per unit length $M$ and length $L$ on normalized frequency $\Omega$, where $\Omega = \omega/(c/L)$. The dashed curve represents the normalized reactance of an acoustic mass $M_A = ML$.

\[
Z(-L) = jz_0kL = j\sqrt{\frac{M}{C}} \omega \sqrt{MCL} = j\omega (ML)
\]

\[
Z'(-L) = \frac{Z(-L)}{z_0} = \frac{j\omega ML}{\sqrt{M/C}} = \frac{j\omega L}{c} = j\Omega
\]
Figure 27: The solid curve represents the dependence of the normalized reactance $X/z_0$ of an acoustic tube with acoustic compliance per unit length $C$ and length $L$ on normalized frequency $\omega/\omega_0$, where $\omega_0 = c/L$. The dashed curve represents the normalized reactance of an acoustic compliance $C_A = CL$. Note that while the reactance of the acoustic compliance is always negative, the reactance of the tube becomes positive for high enough frequency.

$$Z(-L) = -j \frac{z_0}{kL} = -j \sqrt{\frac{M}{C}} \frac{L}{\omega \sqrt{MC}} = \frac{1}{j\omega(CL)}$$

$$\frac{Z(-L)}{z_0} = \frac{1}{j\omega LC} \sqrt{\frac{C}{M}} = \frac{1}{j\omega L} \frac{1}{\sqrt{MC}} = \frac{1}{j\Omega}$$
Figure 28: Prototypical Helmholtz Resonator.

\[
\sin \left( \frac{2\pi f_0 L}{c} \right) \approx \frac{2\pi f_0 L}{c},
\]

\[
\cos \left( \frac{2\pi f_0 L}{c} \right) \approx 1.
\]

\[
C_A = \frac{A_B L_B}{\rho_0 c^2},
\]

\[
M_A = \frac{\rho_0 L_F}{A_F},
\]

\[
f_0 = \frac{1}{2\pi} \frac{1}{\sqrt{M_A C_A}} = \frac{c}{2\pi} \sqrt{\frac{A_F/A_B}{L_F L_B}}.
\]
Figure 29: Prototypical Helmholtz Resonator.
Figure 30: Prototypical Helmholtz Resonator.
Figure 31: Prototypical Helmholtz Resonator.
2 Systems Composed of Two Tube Segments.

Figure 32: Acoustic system composed of two tube segments.
Nearly Uncoupled Tubes

Figure 33: The case of two nearly uncoupled tubes with $A_F \gg A_B$.

Figure 34: Illustrating the solution to Eq. ?? in the case $L_B = 5$ cm, $L_F = 12$ cm, $A_B/A_F = 0.1$. Dotted curves represent the right side of the equation, solid curves the left sides. Asymptotes are indicated by dashed vertical lines. Filled squares axis indicate the natural frequencies of the two-tube system.
Figure 35: The case of two nearly uncoupled tubes with $A_F \ll A_B$.

**Helmholtz Resonance**

\[ Z_B + Z_F = 0 \]

\[ Z_B = -j \frac{\rho_0 c}{A_B} \cot k L_B \approx -j \frac{\rho_0 c}{A_B} \frac{1}{k L_B} \]

\[ Z_F = j \frac{\rho_0 c}{A_F} \tan k L_F \approx j \frac{\rho_0 c}{A_F} k L_F \]

\[ k^2 = \frac{A_F}{L_F L_B A_B} \]

\[ f = \frac{c}{2\pi} \sqrt{\frac{A_F / A_B}{L_F L_B}} \]
2.1 The General Case.

\[
\frac{P}{U_S} = Z = \frac{Z_B Z_F}{Z_B + Z_F}
\]

\[Z_B + Z_F = 0\]

**Z\(_B\)**: impedance of a closed tube of length \(L_B\) & area \(A_B\).

\[Z_B = -j \frac{\rho_0 c}{A_B} \cot kL_B\]

**Z\(_F\)**: impedance of an open tube of length \(L_F\) & area \(A_F\).

\[Z_F = j \frac{\rho_0 c}{A_F} \tan kL_F\]

\[\frac{1}{A_B} \cot kL_B = \frac{1}{A_F} \tan kL_F\]

\[
\cot kL_B = \frac{A_B}{A_F} \tan kL_F
\]
Figure 37: Illustrating the solution to Eq. ?? in the case $L_B = 5 \text{ cm}$, $L_F = 12 \text{ cm}$, $A_B/A_F = 3$. Dotted curves represent the right side of the equation, solid curves the left sides. Asymptotes are indicated by dashed vertical lines. Crosses on the Frequency axis indicate the natural frequencies of the two-tube system.
Figure 38: Illustrating the solution to Eq. ?? in the case $L_B = 5$ cm, $L_F = 12$ cm, $A_B/A_F = 0.333$. Dotted curves represent the right side of the equation, solid curves the left sides. Asymptotes are indicated by dashed vertical lines. Crosses on the Frequency axis indicate the natural frequencies of the two-tube system.
Figure 39: Illustrating the solution to Eq. ?? in the case \( L_B = 12 \) cm, \( L_F = 5 \) cm, \( A_B/A_F = 3 \). Dotted curves represent the right side of the equation, solid curves the left sides. Asymptotes are indicated by dashed vertical lines. Crosses on the Frequency axis indicate the natural frequencies of the two-tube system.
Figure 40: Illustrating the solution to Eq. ?? in the case $L_B = 12 \text{ cm}, L_F = 5 \text{ cm}, A_B/A_F = 0.333$. Dotted curves represent the right side of the equation, solid curves the left sides. Asymptotes are indicated by dashed vertical lines. Crosses on the Frequency axis indicate the natural frequencies of the two-tube system.
Figure 41: Illustrating the solution to Eq. ?? Solid curves represent the left side of the equation, with asymptotes indicated by dotted vertical lines. Dashed curves represent the right side of the equation, with wider dashed vertical lines indicating the asymptotes. Filled squares on the Frequency axis indicate the natural frequencies of the two-tube system. The upper panels display results for $L_B = 5$ cm, $L_F = 12$ cm. The lower panels display results for $L_B = 12$ cm, $L_F = 5$ cm. The left panels display results for $A_B/A_F = 3.0$; The right panels display results for $A_B/A_F = 0.33$;
Table 1: Estimated natural frequencies for the systems analyzed in Fig. 41. Lengths are given in cm, frequencies in Hz.

<table>
<thead>
<tr>
<th>$L_B$</th>
<th>$L_F$</th>
<th>$A_B/A_F$</th>
<th>$F_1$</th>
<th>$F_2$</th>
<th>$F_3$</th>
<th>$F_4$</th>
<th>$F_5$</th>
</tr>
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<tr>
<td>5</td>
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<td>2680</td>
<td>3490</td>
<td>4410</td>
</tr>
<tr>
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<td>0.33</td>
<td>620</td>
<td>1590</td>
<td>2360</td>
<td>3555</td>
<td>4700</td>
</tr>
<tr>
<td>12</td>
<td>5</td>
<td>3.00</td>
<td>350</td>
<td>1460</td>
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<td>0.33</td>
<td>620</td>
<td>1590</td>
<td>2360</td>
<td>3555</td>
<td>4700</td>
</tr>
</tbody>
</table>

$$\cot y = \cos y / \sin y$$

$$\cos kL_B \cos kL_F = \frac{A_B}{A_F} \sin kL_B \sin kL_F$$

If $k$ satisfies this equation when $L_B = X$ and $L_F = Y$, $k$ satisfies this equation when $L_B = Y$ and $L_F = X$. 
\[ \cos kL = \frac{A_B - A_F}{A_B + A_F} \cos k(L_B - L_F) \]

When \( L_B = L_F = L/2 \) the natural frequencies are:

\[ f = \frac{c}{2\pi L} \arccos \frac{A_B - A_F}{A_B + A_F} \]

Figure 42: Solution for the natural frequencies for an \( L = 17 \text{ cm} \) two tube system in the case of \( A_B/A_F = 3.0 \) (dotted line) and \( A_B/A_F = 0.33 \) (dashed line) when \( L_B = L_F = L/2 = 8.5 \text{ cm} \).
Figure 43: Illustrating the dependence of the solution to Eq. ?? on $A_F/A_B$ in the case $L_B = 5$ cm, $L_F = 12$ cm. The solid circles on the right correspond to the limiting cases described by Eqs. ?? and ??, The solid circles on the left correspond to the limiting cases described by Eqs. ?? and ??, The dotted curve corresponds to the Helmholtz Resonance (Eq. ??).
Figure 44: Illustrating the dependence of the solution to Eq. ?? on $L_B$ and $L_F$ in the case $A_B/A_F = 0.33$. 

$L_B$  

$[L_F + L_B = 17 \text{ cm}]$
Figure 45: Illustrating the dependence of the solution to Eq. ?? on \(L_B\) and \(L_F\) in the case \(A_B/A_F = 3.00\).
Figure 46: First two natural frequencies of the two-tube acoustic system of Fig. 32 as a function of the length of the back cavity, $L_B$ for various area ratios $A_B/A_F$. The area ratio $A_B/A_F = 0$ corresponds to no acoustic coupling between the tubes. The total length $L = L_B + L_F = 17$ cm. (Adapted from Stevens, 1999).

\[ f'_0 = f_0 \left(1 + \frac{2}{\pi} \sqrt{\frac{A_B}{A_F}}\right) \]
3 Internal Sources.

![Circuit representation of a source within an acoustic tube system.](image)

Figure 50: Circuit representation of a source within an acoustic tube system.

\[
\frac{U_F}{U_S} = \frac{Z_B}{Z_B + Z_F}
\]

\[
\frac{U_O}{U_F} = S_{21}
\]

\[
\frac{U_O}{U_S} = \frac{U_O U_F}{U_F U_S} = S_{21} \frac{Z_B}{Z_B + Z_F}
\]

The zeroes of \( U_O/U_S \):

frequencies for which \( S_{21} = 0 \) or \( Z_B = 0 \).
3.1 Source in a Uniform Tube

\[ Z_B = -jz_0 \cot kL_B \]
\[ Z_F = jz_0 \tan kL_F \]
\[ S_{21} = \frac{1}{\cos kL_F} \]

\[ \frac{U_O}{U_S} = \frac{1}{\cos kL_F} \times \frac{\cot kL_B}{\cot kL_B - \tan kL_F} \]
\[ \frac{U_O}{U_S} = \frac{\cos kL_B}{\cos kL} \]
4 Perturbations

Increase $A$ by $\Delta A$

\[ A + \Delta A : \quad M_A = \frac{\rho_0 L}{(A + \Delta A)} \approx \frac{\rho_0 (L - \Delta L)}{A} \]

provided

\[ \Delta L = L \frac{\Delta A}{A}. \]

Increase $A$ by $\Delta A$

\[ A + \Delta A : \quad C_A = \frac{(A + \Delta A) L}{\rho_0 c^2} \approx \frac{A (L + \Delta L)}{\rho_0 c^2} \]

provided

\[ \Delta L = L \frac{\Delta A}{A}. \]
Figure 51: Circuit representation of an acoustic circuit with an acoustic mass of $M$ and an acoustic compliance of $C$.

\[ u(t) = U \sin 2\pi f_0 t \]
\[ p_M(t) = P_M \cos 2\pi f_0 t \]
\[ p_M(t) = M_A \frac{du(t)}{dt} \]
\[ P_M = 2\pi f_0 M_A U. \]
\[ \langle E_M \rangle = \frac{1}{4} M_A U^2 \]

\[ \langle E_C \rangle = \frac{1}{4} C_A P_M^2 = \frac{1}{4} C_A (2\pi f_0 M_A U)^2 \]
\[ = \frac{1}{4} C_A \frac{4\pi^2}{(2\pi \sqrt{M_A C_A})^2} M^2 U^2 = \frac{1}{4} M U^2 = \langle E_M \rangle \]
The increase in acoustic mass increases $\langle E_M \rangle$.

$$\langle E_M \rangle = \frac{1}{4} (M_A + \Delta M_A) U^2$$

To still be at a natural frequency $\langle E_C \rangle$ must increase

$$\langle E_C \rangle = \frac{1}{4} C_A P_C^2$$

Increasing $P_C$ at constant $U$ requires that the magnitude of the impedance of the compliance, $1/(2\pi f C_A')$ to increase, which in turn means that the natural frequency must decrease, so

$$f'_0 < f_0$$
4.1 Small Hole in the Closed End

\[ M_H = \frac{\rho_0 L_H}{A_H} \]

\[ C_A = \Delta L \frac{A_T}{\rho_0 c^2} \]

The impedance becomes infinite when:

\[ 0 = \frac{1}{j\omega C_A} + j\omega M_H \]

\[ = \frac{1}{j\omega \frac{A_T}{\rho_0 c^2} \Delta L} + j\omega \rho_0 \frac{L_H}{A_H} \]

or when

\[ \Delta L = \left(\frac{c}{\omega}\right)^2 \frac{A_H}{A_T} \frac{1}{L_H} \]

The effect of the hole is to shorten the tube by \( \Delta L \) and to raise the natural frequencies of the tube.
4.2 General Case

Figure 53: Illustrating a perturbation in a uniform tube that is closed at one end and opened at the other. The tube is of length $L$ and cross-sectional area $A$. The perturbation in the tube consists of a constriction which reduces the area of the tube to $A_C$ at a distance of $L_C$ from the closed end of the tube.
Figure 54: The solid curve represents the amplitude of pressure ($|P(x)|$) and dotted curve represents the amplitude of volume velocity ($|U(x)|$) within a tube closed at one end and open at the other.
Figure 55: Curves showing the relative direction and magnitude of the change in the lowest three natural frequencies ($F_1$, $F_2$, and $F_3$) for a uniform tube when the cross-sectional area is reduced at a point along the tube. A − indicates that the natural frequency is reduced, a + indicates that the natural frequency is increased.
6 Vowel Production

Figure 59: Illustrating the dependence of $F_1$ and $F_2$ on different shapes of acoustic tubes. The midpoint corresponds to the natural frequencies of a uniform tube of length 15.4 cm ($F_1 = 556$ and $F_2 = 1670$ Hz). From Stevens (1999).
6.1 $F_1$ - First Formant Frequency

Figure 60: Vocal tract shape used by an adult male speaker to produce the high vowels /i/ (left) and /u/ (right). Adapted from Perkell (1969).
High Vowels

Figure 61: Spectra of the synthetic vowels /i/ (left panels), /u/ (center panels), and /ah/ (right panels) with formant frequencies appropriate for an adult female (upper panels) and male (lower panels) talkers. The time waveforms and time windows used in the spectral analysis are also shown. (Adapted from Klatt and Klatt, 1990).
Figure 62: Spectrograms of 100 ms portions of the three vowels in Fig. 61. (From Klatt and Klatt, 1990.)
Non-High Vowels

Figure 63: Vocal tract shapes used by an adult male speaker of English to produce the low vowels /a/ (left) and /ae/ (right). (Adapted from Perkell, 1969.)
Low Vowels

Figure 64: Vocal tract shapes used by an adult male speaker of French to produce the high vowels /e/ (left) and /o/ (right). (Adapted from Bothorel et al., 1986).
6.2 $F_2$ - Second Formant Frequency

Figure 65: Left: vocal tract shapes used by an adult male speaker to produce the low vowels /a/ and /æ/. Right: Two tube model for the production of low vowels. (Adapted from Perkell, 1971, and Stevens, 1999.)
Figure 66: First four natural frequencies for the two-tube configuration of Fig. 65 as a function of the length ($L_B$) of the back section. The total length $L = L_B + L_F = 16$ cm and $A_F = 3$ sq cm. Dashed curves correspond to $A_B = 0$, solid curves to $A_B = 0.5$ sq cm. (Adapted from Stevens, 1989.)
High Vowels

Figure 67: Three-tube model of the vocal tract shape used by an adult male speaker to produce the high vowels /i/ and /u/. (Adapted from Stevens, 1989.)

Figure 68: The middle panel: the natural frequencies of the three-tube configuration of Fig. 67 as a function of $L_B$, with $L_C = 5$ and $L_B + L_C + L_F = 16$ cm. The cross-sectional areas assumed in the computation were $A_1 = 3$, $A_C = 0.3$, and $A_F = 1.0$ sq cm. The portions of the curves corresponding to non-low front vowels ($L_B = 6$ and 11 cm) are marked by vertical lines. Spectra corresponding to $L_B = 8$ and 10 cm are shown. (Adapted from Stevens, 1999.)
6.3 Perturbation Theory

Figure 69: Curves showing the relative direction and magnitude of the change in the lowest three natural frequencies ($F_1$, $F_2$, and $F_3$) for a uniform tube when the cross-sectional area is reduced at a point along the tube. A $-$ indicates that the natural frequency is reduced, a $+$ indicates that the natural frequency is increased.