Quiz 1 Wed 10/16 7:30 pm
in 32-124, 32-144, 34-101 (see website for guidance)
May bring one 8 1/2” x 11” “cheat sheet” (otherwise closed book)
Coverage up through L10 (Hashing III)
*No lecture Tuesday; quiz review Wed at recitation times (see website)
Readings: Newton’s Method (Wikipedia)

- High-Precision multiplication (Karatsuba)
- Computing \( \sqrt{2} \)
- Newton’s Method — in general
  - Computing \( \sqrt{a} \)
  - Division
High-precision multiplication (of integers)

Multiplying two n-digit #’s (radix=r, e.g. r=2 or r=10)

\[ 0 \leq x, y \leq r^n \]

\[ x = x_1 \cdot r^{n/2} + x_0 \]

\[ y = y_1 \cdot r^{n/2} + y_0 \]

\[ z = x \cdot y = x_1 y_1 \cdot r^n + (x_0 y_1 + x_1 y_0) r^{n/2} + x_0 y_0 \]

\[ \Rightarrow T(n) = 4T(n/2) + \Theta(n) = \Theta(n^2) \]

Karatsuba uses only 3 recursive calls:

\[ z_0 = x_0 \cdot y_0 \]

\[ z_2 = x_1 \cdot y_1 \]

\[ z_1 = (x_0 + x_1) \cdot (y_0 + y_1) - z_0 - z_2 = x_0 y_1 + x_1 y_0 \]

\[ z = z_2 \cdot r^n + z_1 \cdot r^{n/2} + z_0 \]

\[ T(n) = 3T(n/2) + \Theta(n) \]

\[ = \Theta(n \log(3)) = \Theta(n^{1.5849625\ldots}) \]

Better than \( \Theta(n^2) \).

Python does this!
Computing $\sqrt{2}$ (to high precision)

$$\sqrt{2} = 1.414 213 562 373 095 048 801 688 724 209 698 078 569 671 875 \ldots$$

How hard is it to compute $10^6$-th digit?

Why should we care?

- "academic interest"
- high precision may be needed in some applications

E.g., Dijkstra's shortest-path algorithm on lattice pts in plane
paths have lengths of form $\sum_i h_i^2$ for integral $h_i^2$

\[ a^2 + b^2 = h^2 \]

How many digits are needed to compare path lengths?

(Unknown!!)

$$\sqrt{1} + \sqrt{40} + \sqrt{60} = 15.070 522 012 75 \ldots$$

$$\sqrt{12} + \sqrt{17} + \sqrt{56} = 15.070 522 014 30 \ldots$$

- Because we can!
- Illustrates nifty design principles for algorithms

(Newton's Method, Divide & Conquer)
**Newton's Method**

Given function \( f(x) \), want to find root of \( f(x) = 0 \) by successive approximation: 

\[
x_0 = \text{initial guess}
\]

\[
x_1 = \text{better guess}
\]

more accuracy

\[
x_2
\]

\[
\vdots
\]

E.g. \( f(x) = x^2 - 2 \) (root at \( \pm \sqrt{2} \))

How to improve a guess?

Tangent at \( (x_i, f(x_i)) \)

is \( y = f(x_i) + f'(x_i) \cdot (x - x_i) \)

\( \text{derivative} \)

\( x_{i+1} = \text{intercept of tangent on x-axis} \)

\[
x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}
\]

For \( f(x) = x^2 - a \)

\[
x_{i+1} = x_i - \frac{(x_i^2 - a)}{2x_i} = \frac{x_i + a/x_i}{2}
\]

= average of \( x_i \) and \( a/x_i \)
Example: \(a = 2\)

- \(x_0 = 1.000\ 000\ 000\)
- \(x_1 = 1.500\ 000\ 000\)
- \(x_2 = 1.411\ 666\ 666\)
- \(x_3 = 1.414\ 215\ 686\)
- \(x_4 = 1.414\ 213\ 562\)

\[
\left(1 + \frac{3}{2}\right)^\frac{1}{2} = 3/2
\]

\[
\left(\frac{3}{2} + \frac{3/2}{2}\right)^\frac{1}{2} = 17/12
\]

\[
\frac{577}{408} \quad \text{fast convergence!}
\]

Error analysis:

Suppose \(x_n = \sqrt{a \cdot (1 + \varepsilon_n)}\)

Then

\[
x_{n+1} = \frac{x_n + (a/x_n)}{2}
\]

\[
= \frac{\sqrt{a \cdot (1 + \varepsilon_n) + (a/(\sqrt{a \cdot (1 + \varepsilon_n)}))}}{2}
\]

\[
= \frac{\sqrt{a \cdot \left( (1 + \varepsilon_n) + 1/(1 + \varepsilon_n) \right)}}{2}
\]

\[
= \sqrt{a \cdot \left( \frac{2 + 2\varepsilon_n + \varepsilon_n^2}{2(1 + \varepsilon_n)} \right)}
\]

\[
= \sqrt{a \cdot \left( 1 + \frac{\varepsilon_n^2}{2(1 + \varepsilon_n)} \right)} = \sqrt{a \cdot (1 + \varepsilon_{n+1})}
\]

where \(\varepsilon_{n+1} = \frac{\varepsilon_n^2}{2(1 + \varepsilon_n)}\)

"quadratic convergence"

# correct digits doubles each step

precision grows exponentially with # of iterations, n

note \(\varepsilon_{n+1} > 0, \quad \text{too...} \quad \text{Converge "from above"}
For d-digit precision: $1.414213562373\ldots$
we want integer $\lfloor 10^d \sqrt{a} \rfloor$
$= \lfloor \sqrt{2 \cdot 10^{2d}} \rfloor = \lfloor \sqrt{a} \rfloor$ for $a = 2 \cdot 10^{2d}$
which we compute using Newton with integer division (floors)

$$x_{n+1} = \left\lfloor \frac{x_n + \lfloor a/x_n \rfloor}{2} \right\rfloor$$

$= \left\lfloor \frac{x_n + a/x_n}{2} \right\rfloor$ (Exercise !)

Knuth, Vol 1, Section 1.2.4
(exercise 35)

Note: $\frac{x+y}{2} \geq \sqrt{xy}$

so $\frac{x + a/x}{2} \geq \sqrt{a}$

and $\left\lfloor \frac{x + a/x}{2} \right\rfloor \geq \left\lfloor \sqrt{a} \right\rfloor$

converge "from above"

Iteration can't "get stuck" at integer values $\lfloor \sqrt{a} \rfloor$, since

$$E_n = \frac{E_{n-1}^2}{2(1+E_{n-1})} < E_{n-1} \text{ if } E_{n-1} < 1$$

for ordinary Newton's Method (without floors), so even
with rounding down (integer divisions) we must reach

$\lfloor \sqrt{a} \rfloor$.
High-precision division:

- want to compute \( a/b \) to high precision
- compute \( 1/b \) to high precision, then mpy by \( a \)
- high precision \( 1/b \) means \( \lfloor R/b \rfloor \)
  
  where, e.g., \( R = 10^d \)

- Use Newton's Method on
  
  \[
  f(x) = \frac{1}{x} - \frac{b}{R} \quad \text{(root at } x = \frac{R}{b})
  \]

(Details next time...)