Admin:

Pset #4 out today
Quizzes being graded

Readings: (see web site for links)

Outline:

☐ Review of last lecture
☐ High-precision division
☐ Cube roots; domains of convergence
☐ Some pretty pictures
Review:

Add/subtract n-digit #3's in **linear time** $O(n)$

Multiply n-digit #5's in time

$O(n \log_2 3)$ [Karatsuba]

$O(n \log n \log \log n)$ [Schönhage-Strassen]

Newton's Method for solving $f(x) = 0$

to get n-digit result

$x_0 \leftarrow \text{(good starting guess)}$

$$x_{i+1} \leftarrow x_i - \frac{f(x_i)}{f'(x_i)}$$

$f'(x) = \frac{d}{dx} f(x)$

Computing $\sqrt{a} = \text{root of } x^2 - a$

# of correct digits doubles with each iteration of

$$x_{i+1} \leftarrow x_i + \frac{(a/x_i)}{2}$$

Note: need to divide
High precision division

- Suppose we want high-precision representation of \( a/b \)
- Idea: Compute high-precision rep of \( 1/b \), then multiply that by \( a \).
- High-precision rep of \( 1/b \) means
  \[ \left\lfloor \frac{R}{b} \right\rfloor \] where \( R \) is large value such it is easy to divide by \( R \)
  (e.g., \( R = 2^k \) for binary reps.)
  (drop low-order \( k \) bits)

Consider: Newton's method for computing \( R/b \):

\[
f(x) = \frac{1}{x} - \frac{b}{R} \quad \text{(Zero at } x = \frac{R}{b})
\]

\[
f'(x) = -\frac{1}{x^2}
\]

\[
x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)} = x_i - \frac{\left(\frac{1}{x_i} - \frac{b}{R}\right)}{-\frac{1}{x^2}}
\]

\[
= x_i + x_i \left(\frac{1}{x_i} - \frac{b}{R}\right) = 2x_i - \frac{bx_i^2}{R}
\]
Example:

\[
\frac{R}{b} = \frac{2^{16}}{5} = \frac{65536}{5} = 13107.2
\]

\[
x_0 = 2^{14} = 16384
\]

\[
x_i = 2 \cdot 16384 - 5 \cdot 16384^2 / 65536 = 12288
\]

\[
x_2 = 2 \cdot 12288 - 5 \cdot 12288^2 / 65536 = 13056
\]

\[
x_3 = 2 \cdot 13056 - 5 \cdot 13056^2 / 65536 = 13107
\]

**Error Analysis:**

\[
x_{i+1} = 2x_i - b x_i^2 / R
\]

Assume \(x_i = \frac{R}{b} (1 + \varepsilon_i)\)

\[
= 2 \frac{R}{b} (1 + \varepsilon_i) - b \frac{R}{b} (\frac{R}{b})^2 (1 + \varepsilon_i)^2
\]

\[
= \frac{R}{b} \left[ (2 + 2 \varepsilon_i) - (1 + 2 \varepsilon_i + \varepsilon_i^2) \right]
\]

\[
= \frac{R}{b} \left( 1 - \varepsilon_i^2 \right) = \frac{R}{b} \left( 1 + \varepsilon_{i+1} \right)
\]

where \(\varepsilon_{i+1} = -\varepsilon_i^2\) "quadratic convergence" (again)

\# digits correct doubles at each step

\# digits correct after \(k\) steps is \(\Theta(2^k)\)

(assuming good starting guess)
- Each iteration requires 2 multiplications
  (plus some adds/subs & shifting)
- Can organize to work with numbers that
  increase (double) in precision at each step.
  \[ M(d) = \text{work to multiply two d-digit #s} \]
  \[ D(d) = \text{time to compute d-digit value} \quad [R/b] \]
  \[ = 2M(d) + \Theta(d) + D(d/a) \]
  \[ = \Theta(M(d)) \]
- Complexity of division \leq complexity of multiplication
  \((\text{other way is also true})\)
Effect of starting pt:

for $\sqrt{a}$: $f(x) = x^2 - a$

$x_{i+1} = \frac{x_i + a/x_i}{2}$

- If $x_0 > 0$: converge to $\sqrt{a}$
- If $x_0 < 0$: converge to $-\sqrt{a}$
- If $x_0 = 0$: blows up!

A simple domain of convergence: 1 pt on boundary.
How about cube roots?

\[ \sqrt[3]{a} : f(x) = x^3 - a \]

\[ x_{i+1} = x_i - \frac{(x_i^3 - a)}{3x_i^2} \]

Unless you start at \( x_0 = 0 \), or hit \( x_i = 0 \) along the way, you'll end up at \( \sqrt[3]{a} \).

e.g. for \( \sqrt[3]{3} \)

\[
\begin{align*}
  x_0 &= -2 \\
  x_1 &= -1.25 \\
  x_2 &= -0.62 \\
  x_3 &= 0.45 \\
  x_4 &= 1.92 \\
  x_5 &= 1.37 \\
  x_6 &= 1.09 \\
  x_7 &= 1.0074
\end{align*}
\]
Let's look at \( f(x) = x^3 - a \) in complex plane.
(and take \( a = 1 \) for simplicity) NM still works...

\[ \pm \frac{1}{2} \pm \frac{\sqrt{3}}{2} \]

\[ 3\sqrt[3]{1} = \{ 1, -\frac{1}{2} \pm \frac{\sqrt{3}}{2} \} \]

What is domain of convergence for
- \[ 1 \]
- \[ -\frac{1}{2} + \frac{\sqrt{3}}{2} \]
- \[ -\frac{1}{2} - \frac{\sqrt{3}}{2} \]

Expect

But this is (mostly) red?

???

(from previous page)
Show figs from lpslop:

Color code

After 1 NM iteration

After many -- beautiful fractal!!!

(Setting it in motion, too...)

URL's:

google images "newton fractal"
en.wikipedia.org/wiki/Newton_Fractal
facstaff.unca.edu/mamcelur/mathematicaGraphics/Newton
www.mitchr.me/55/newton
vimeo.com/11556693 (rotating newton)