Lecture 21

Distributed Algorithms

- model (see slides)
- algorithms
  - Leader Election < impossibility?
  - BFS (see slides)
  - Maximal independent set
Ring Leader Election [LeLann] [Chang, Roberts]

- Initialize \( \text{send-max} \leftarrow \text{UID} \)

- Repeat until find leader:
  - send \( \text{send-max} \) clockwise
  - receive \( \text{incoming-max} \)
  - if \( \text{incoming-max} > \text{send-max} \)
    - \( \text{send-max} \leftarrow \text{incoming-max} \)
  - else if \( \text{incoming-max} \leq \text{send-max} \)
    - do nothing
  - else if \( \text{incoming-max} = \text{send-max} = \text{UID} \)
    - output "I am leader"

Leader Election: [LCR] algorithm
Maximal Independent Set

Given graph $G = (V, E)$

$$\text{def } I \subseteq V \text{ "independent" if } \forall u, v \in I \quad (u, v) \notin E$$

"maximal independent set" if $\forall I' \text{ independent, } I \neq I'$

Maximal Independent set is NP-complete hard

Maximal Independent set solved by greedy algorithm

Note input is the interconnection graph of the network!

MIS has $O(\log n)$ round protocol

Here: special case Graph has degrees $\leq d$

will show $O(d \log n)$ round algorithm
(Simplified) Luby's Algorithm

Initialize

all nodes added to "live" set
MLS <- ∅

Phase

1) each live node "marks" self with \( p = \frac{1}{2d} \)

2) each marked node checks nbrs.
   if any marked, \( v \) "unmarks" itself

3) each remaining marked node:
   - adds itself to MLS
   - removes itself & all neighbors from "live" set

Thm Probability that number of phases > \( 8d \ln n \)
   is at most \( \frac{1}{n} \)

Corr Expected number of phases is \( O(d \ln n) \)
Example:

\[ \text{Example:} \quad \begin{array}{c}
\text{unmark} \\
\Rightarrow \\
\Rightarrow \\
\Rightarrow \\
\text{empty graph}
\end{array} \]

Main Lemma \quad \Pr [v \text{ live } \Rightarrow v \text{ adds self to MIS in one round}] \geq \frac{1}{4d}

\begin{align*}
\Pr & v \text{ live } \\
& \Pr [v \text{ marks self in step 1}] = \frac{1}{2d} \\
& \Pr [\text{any nbr of } v \text{ marks self in step 1}] \leq \sum_{w, \text{ neighbor of } v} \frac{1}{2d} \quad \text{(union bound)} \\
& \leq 2d \cdot \frac{1}{2d} = \frac{1}{2} \quad \text{(all degrees } \leq d) \\
\therefore \quad & \Pr [v \text{ marks self } \Rightarrow \text{ stays marked after step 2}] = \frac{1}{2d} \cdot \frac{1}{2} = \frac{1}{4d} \quad \blacksquare
\end{align*}
Proof of Thm.

Idea: at each phase \( \frac{1}{4d} \) fraction of nodes deleted

\[
\Pr[\text{v stays alive after } c_4 d \ln n \text{ rounds}] \leq (1 - \frac{1}{4d})^{c_4 d \cdot \ln n} \leq e^{-c \ln n} = n^{-c}.
\]

\[
\Pr[\text{any v stay alive after } 8d \ln n \text{ rounds}] \leq n \cdot n^{-2} = \frac{1}{n}.
\]

Slight modification of algorithm, bigger modification of analysis gives \( O(\log n) \) rounds for general graphs. "Luby's Algorithm"

- Can be made asynchronous
- Needs no UID's
Applications

- Maximal Matching -
  matching where no edge can be added
  (might not be maximum!)

Can use MIS to find one:

\[ G = (V, E) \implies G' = (V', E') \]

\[ V' \subseteq E \]
\[ E' \subseteq \text{nodes in } V' \text{ whose respective edges in } G \text{ are adjacent} \]

Maximal Matching \( G \)
MIS in \( G' \)

- Vertex coloring -
  can color nodes of graph with \( \leq (\text{max degree} + 1) \) colors