Assume a single tap model with \( G_{0,m} = G_m \).
Assume \( G_m \) is circ. symmetric Gaussian with 
\[ \mathbb{E}[|G_m|^2] = 1. \]

The magnitude is Rayleigh with

\[ f_{|G_m|}(|g|) = 2|g| \exp\{-|g|^2\} \quad ; \quad |g| \geq 0 \]

\[ \Re(G_m) \sim \mathcal{N}(0,1/2); \quad \Im(G_m) \sim \mathcal{N}(0,1/2) \]
\( V_m = U_m G_m + Z_m; \quad \Re(Z_m), \Im(Z_m) \sim \mathcal{N}(0, WN_0/2) \)

Antipodal binary communication does not work here. It can be viewed as phase modulation \((180^\circ)\); the phase of \( V_m \) is independent of \( U_m \).

We could use binary modulation with \( U_m=0 \) or \( U_m=a \); this is awkward and performs badly.

Consider pulse position modulation over 2 samples.

\[
\begin{align*}
H = 0 & \quad \rightarrow \quad (U_0, U_1) = (a, 0) \\
H = 1 & \quad \rightarrow \quad (U_0, U_1) = (0, a).
\end{align*}
\]
This is equivalent to any binary scheme which uses one or the other of 2 symmetric complex degrees of freedom.

\[ H = 0 \quad \rightarrow \quad V_0 = aG_0 + Z_0; \quad V_1 = Z_1 \]
\[ H = 1 \quad \rightarrow \quad V_0 = Z_0; \quad V_1 = aG_1 + Z_1. \]

\[ H = 0 \quad \rightarrow \quad V_0 \sim \mathcal{N}_c(0, a^2 + WN_0); \quad V_1 \sim \mathcal{N}_c(0, WN_0) \]
\[ H = 1 \quad \rightarrow \quad V_0 \sim \mathcal{N}_c(0, WN_0); \quad V_1 \sim \mathcal{N}_c(0, a^2 + WN_0). \]

\( \mathcal{N}_c(0, \sigma^2) \) means iid real, imaginary, each \( \mathcal{N}(0, \sigma^2/2) \).
\[ H = 0 \quad \rightarrow \quad V_0 \sim \mathcal{N}_c(0, a^2+WN_0); \quad V_1 \sim \mathcal{N}_c(0, WN_0) \]

\[ H = 1 \quad \rightarrow \quad V_0 \sim \mathcal{N}_c(0, WN_0); \quad V_1 \sim \mathcal{N}_c(0, a^2+WN_0). \]

\[ f(v_0, v_1|H=0) = \alpha \exp \left\{ - \frac{|v_0|^2}{a^2 + WN_0} - \frac{|v_1|^2}{WN_0} \right\} \]

\[ f(v_0, v_1|H=1) = \alpha \exp \left\{ - \frac{|v_0|^2}{WN_0} - \frac{|v_1|^2}{a^2 + WN_0} \right\} \]

\[ LLR(v_0, v_1) = \ln \left\{ \frac{f(v_0, v_1|H_0)}{f(v_0, v_1|H_1)} \right\} = \frac{[|v_0|^2 - |v_1|^2] a^2}{(a^2 + WN_0)(WN_0)} \]

**Given** \(H = 0\), \(|V_0|^2\) **is exponential, mean** \(a^2+WN_0\)

and \(|V_1|^2\) **is exponential, mean** \(WN_0\). **Error if**

the sample value for the first less than that of the second.
Let $X_0 = |V_0|^2$ and $X_1 = |V_1|^2$. Given $H_0$, $X_1$ is an exponential rv with mean $WN_0$ and $X_0$ is exponential with mean $WN_0 + a^2$.

$$Pr(e | H = 0) = Pr(X_0 < X_1 | H = 0).$$

The text use tedious calculation to show that

$$Pr(e | H_0) = \frac{WN_0}{a^2 + 2WN_0} = \frac{1}{2 + a^2/(WN_0)}$$

More elegant approach: consider a Poisson process with arrivals at rate $\lambda$. Segment time into tiny intervals of width $\delta$. The probability of an arrival in an interval is $\delta \lambda$; arrivals are independent.
The probability of no arrival for \( n \) intervals is
\[
(1 - \delta \lambda)^n = \exp(n \ln(1 - \delta \lambda)) \approx \exp(-n\delta \lambda).
\]

The probability of no arrival for an interval \( y \) is \( \exp(-\lambda y) \). The time to the first arrival is an exponential rv \( Y_0 \) with \( f_{Y_0}(y) = \lambda \exp(-\lambda y) \).

Consider another independent Poisson process with rate \( \gamma \). The time to the first arrival here is exponential with \( f_{Y_1}(y) = \gamma \exp(-\gamma y) \).

The sum of these processes is Poisson with rate \( \lambda + \gamma \). Each arrival is from the first process with probability \( \lambda/\lambda + \gamma \). Thus
\[
\Pr(Y_0 < Y_1) = \frac{\lambda}{\lambda + \gamma}
\]
\[
\Pr(Y_0 < Y_1) = \frac{\lambda}{\lambda + \gamma} = \frac{1}{1 + \gamma/\lambda}
\]

For the exponential rv’s \(X_0\) and \(X_1\) conditional on \(H=0\) in the Rayleigh fading case, \(\lambda = 1/(a^2 + WN_0)\) and \(\gamma = 1/(WN_0)\). Thus

\[
\Pr(X_0 < X_1|H=0) = \frac{1}{1 + \frac{a^2 + WN_0}{WN_0}} = \frac{1}{2 + a^2/(WN_0)}
\]

This is the probability of error given \(H=0\), and by symmetry, the probability of error given \(H=1\). The signal power is \(a^2/2\) and there are \(W/2\) bits per second. Thus \(E_b = a^2/W\) and

\[
\Pr(e) = \frac{1}{2 + E_b/N_0}
\]
We next look at non-coherent detection. We use the same model except to assume that 
\[ |g_0| = |g_1| = \tilde{g} \] is known. We calculate \( \Pr(e) \) conditional on \( \tilde{g} \).

We find that knowing \( \tilde{g} \) does not aid detection. We also find that the Rayleigh fading result occurs because of the fades rather than lack of knowledge about them.

\[
\begin{align*}
H = 0 & \quad \rightarrow \quad V_0 = a\tilde{g}e^{i\phi_0} + Z_0; \quad V_1 = Z_1 \\
H = 1 & \quad \rightarrow \quad V_0 = Z_0; \quad V_1 = a\tilde{g}e^{i\phi_1} + Z_1.
\end{align*}
\]

The phases are independent of \( H \), so \( |V_0| \) and \( |V_1| \) are sufficient statistics.

The ML decision is \( \tilde{H} = 0 \) if \( |V_0| \geq |V_1| \). This decision does not depend on \( \tilde{g} \).
Since the phase of $G$ and that of the noise are independent, we can choose rectangular coordinates with real $G$. The calculation is straightforward but lengthy.

$$
Pr(e) = \frac{1}{2} \exp\left(\frac{-a^2\tilde{g}^2}{2WN_0}\right) = \frac{1}{2} \exp\left(\frac{-E_b}{2N_0}\right)
$$

If the phase is known at the detector,

$$
Pr(e) = Q\left(\frac{a^2\tilde{g}^2}{WN_0}\right) \leq \sqrt{\frac{N_0}{2\pi E_b}} \exp\left(\frac{-E_b}{2N_0}\right)
$$

When the exponent is large, the db difference in $E_b$ to get equality is small.
If $Pr(e)$ (incoherent result) is averaged over fading, we get flat Rayleigh fading result.

Rayleigh fading result and incoherent result based on sending one bit at a time and using 2 complex degrees of freedom to send that bit.

For that case, knowing the channel amplitude does not help (the incoherent receiver does not use it).

Knowing the channel phase helps a little but not much.
Diversity

Consider a two tap model. More generally consider independent observations of the input.

Consider the input $H=0 \rightarrow a, 0, 0, 0$ and $H=1 \rightarrow 0, 0, a, 0$.

For $H=0$, $\vec{V}' = aG_{0,0}, aG_{1,1}, 0, 0$. For $H=1$, $\vec{V}' = 0, 0, aG_{0,2}, aG_{1,3}$. 
Assume each \( G \) is \( \mathcal{N}_c(0, 1) \) and each \( Z \) is \( \mathcal{N}_c(0, \sigma^2) \).

Given \( H = 0 \), \( V_1 \) and \( V_2 \) are \( \mathcal{N}_c(0, a^2 + \sigma^2) \) and \( V_2, V_3 \) are \( \mathcal{N}_c(0, \sigma^2) \).

Given \( H = 1 \), \( V_1 \) and \( V_2 \) are \( \mathcal{N}_c(0, \sigma^2) \) and \( V_2, V_3 \) are \( \mathcal{N}_c(0, a^2 + \sigma^2) \).

Sufficient statistic is \( |V_j|^2 \) for \( 1 \leq j \leq 4 \). Even simpler, \( |V_1|^2 + V_2|^2 - |V_3|^2 - |V_4|^2 \) is a sufficient statistic.

\[
Pr(e) = \frac{4 + 3\frac{a^2}{\sigma^2}}{\left(2 + \frac{a^2}{\sigma^2}\right)^3} = \frac{4 + \frac{3E_b}{2N_0}}{\left(2 + \frac{E_b}{2N_0}\right)^3}
\]

This goes down with \( (E_b/N_0)^{-2} \) as \( (E_b/N_0) \to \infty \).
CHANNEL MEASUREMENT

Channel measurement is not very useful at the receiver for single bit transmission in flat Rayleigh fading.

It is useful for modifying transmitter rate and power.

It is useful when diversity is available.

It is useful if a multitap model for channel is appropriate. This provides a type of diversity (each tap fades approximately independently).

Diversity results differ greatly depending on whether receiver knows channel and transmitter knows channel.
**SIMPLE PROBING SIGNALS**

Assume $k_0$ channel taps, \( G_{0,m}, \ldots, G_{k_0-1,m} \).

\[
V'_m = u_m G_{0,m} + u_{m-1} G_{1,m} + \cdots + u_{m-k_0+1} G_{k_0-1,m}
\]

Send \((a, 0, 0, \ldots, 0)\)

\[
\vec{V}' = (aG_{0,0}, aG_{1,1}, \ldots, aG_{k_0-1,k_0-1}, 0, 0, \ldots, 0)
\]

\[
V_m = V'_m + Z_m. \text{ Estimate } G_{m,m} \text{ as } V_m/a. \text{ Estimation error is } \mathcal{N}_c(0, WN_0/a^2).
\]
Pseudonoise (PN) PROBING SIGNALS

A PN sequence $\vec{u}$ is a binary sequence that appears to be iid. It is generated by a binary shift register with the mod-2 sum of given taps fed back to the input. With length $k$, it generates all $2^k - 1$ binary non-zero $k$-tuples and is periodic with length $2^k - 1$.

\[
\begin{align*}
\text{PN sequence} & \quad 0 \rightarrow a, 1 \rightarrow -a \\
& \quad \overrightarrow{G} \quad \overrightarrow{V}' \quad \overrightarrow{V} \quad \tilde{u} \quad \overrightarrow{G} + \Psi \\
\tilde{u} & \text{ is } \approx \text{ orthogonal to each shift of } \tilde{u} \text{ so } \\
\sum_{m=1}^{n} u_m u_{m+k}^* & \approx \begin{cases} 
a^2 n & ; \ k = 0 \\
0 & ; \ k \neq 0 \end{cases} = a^2 n \delta_k \\
\text{If } \tilde{u} & \text{ is matched filter to } \tilde{u}, \text{ then } \tilde{u} \ast \tilde{u} = a^2 n \delta_j.
\end{align*}
\]
Binary feedback shift register

Periodic with period $15 = 2^4 - 1$
If $\tilde{u} \ast \tilde{u} = a^2 n \delta_j$, then

$$\tilde{V}' \ast \tilde{u} = (\tilde{u} \ast \tilde{G}) \ast \tilde{u} = (\tilde{u} \ast \tilde{u} \ast \tilde{G}) = a^2 n \tilde{G}$$

The PN property has the same effect as using a single input surrounded by zeros.

The response at time $m$ of $\tilde{u}$ to $\tilde{Z}$ is the sum of $n$ iid rv's each of variance $a^2 N_0 W$.

The sum has variance $a^2 n N_0 W$. After scaling by $1/(a^2 n)$, $E[|\psi_k|^2] = \frac{N_0 W}{a^2 n}$.

The output is a ML estimate of $\tilde{G}$; MSE decreases with $n$. 

17
RAKE RECEIVER

The idea here is to measure the channel and make decisions at the same time.

Assume a binary input, $H=0 \rightarrow \vec{u}^0$ and $H=1 \rightarrow \vec{u}^1$

With a known channel $\vec{g}$, the ML decision is based on pre-noise inputs $\vec{u}^0 * \vec{g}$ and $\vec{u}^1 * \vec{g}$.

$$\Re(\langle \vec{v}, \vec{u}^0 * \vec{g} \rangle) \geq \Re(\langle \vec{v}, \vec{u}^1 * \vec{g} \rangle).$$

$\tilde{H}=0$

$\tilde{H}=1$

We can detect using filters matched to $\vec{u}^0 * \vec{g}$ and $\vec{u}^1 * \vec{g}$
Note the similarity of this to the block diagram for measuring the channel.

If the inputs are PN sequences (which are often used for spread spectrum systems), then if the correct decision can be made, the output of the corresponding arm contains a measurement of $\vec{g}$. 
$\vec{u}^1$ and $\vec{u}^0$ are non-zero from time 1 to $n$. $\vec{v}'$ is non-zero from 1 to $n+k_0-1$.

$\tilde{\vec{u}}^1$ and $\tilde{\vec{u}}^0$ are non-zero from $-n$ to $-1$ (receiver time).

If $H = 1$ or $H = 0$, then $\vec{g}$ plus noise appears from time 0 to $k_0 - 1$ where shown. Decision is made at time 0, receiver time.
If $\hat{H} = 0$, then a noisy version of $\vec{g}$ probably exists at the output of the matched filter $\vec{u}^0$. That estimate of $\vec{g}$ is used to update the matched filters $\tilde{\vec{g}}$.

If $T_c$ is large enough, the decision updates can provide good estimates.
Suppose there is only one Rayleigh fading tap in the discrete-time model.

Suppose the estimation works perfectly and \( \bar{g} \) is always known. Then the probability of error is the coherent error probability \( Q(\sqrt{E_b/N_0}) \) for orthogonal signals and \( E_b = a^2 n |g|^2 / W \).

This is smaller than incoherent \( \Pr(e) = \frac{1}{2} \exp\{-E_b/(2N_0)\} \).

Averaging over \( G \), incoherent result is \( \frac{1}{2+{E_b/N_0}} \) and coherent result is at most half of this.

Measurement doesn’t help here.
Diversity

Consider a two tap model. More generally consider independent observations of the input.

Consider the input $H=0 \rightarrow a, 0, 0, 0$ and $H=1 \rightarrow 0, 0, a, 0$.

For $H=0$, $V'_m = aG_{0,0}, aG_{1,1}, 0, 0$. For $H=1$, $V'_m = 0, 0, aG_{0,2}, aG_{1,3}$. 

\[ \text{Input } \{a, 0\} \rightarrow U_m \rightarrow U_{m-1} \]

\[ \begin{align*}
G_{0,m} & \\
G_{1,m} & \sum V_m' + Z_m \rightarrow V_m
\end{align*} \]
Assume each $G$ is $\mathcal{N}_c(0, 1)$ and each $Z$ is $\mathcal{N}_c(0, \sigma^2)$.

Given $H=0$, $V_1$ and $V_2$ are $\mathcal{N}_c(0, a^2 + \sigma^2)$ and $V_2, V_3$ are $\mathcal{N}_c(0, \sigma^2)$.

Given $H=1$, $V_1$ and $V_2$ are $\mathcal{N}_c(0, \sigma^2)$ and $V_2, V_3$ are $\mathcal{N}_c(0, a^2 + \sigma^2)$.

Sufficient statistic is $|V_j|^2$ for $1 \leq j \leq 4$. Even simpler, $|V_1|^2 + |V_2|^2 - |V_3|^2 - |V_4|^2$ is a sufficient statistic.

$$\Pr(e) = \frac{4 + 3 \frac{a^2}{\sigma^2}}{(2 + \frac{a^2}{\sigma^2})^3} = \frac{4 + \frac{3E_b}{2N_0}}{(2 + \frac{E_b}{2N_0})^3}$$

This goes down with $(E_b/N_0)^{-2}$ as $(E_b/N_0) \to \infty$.  
