Midterm

• You have 90 minutes to complete the quiz.

• This is an open-book quiz. You are allowed to have a double-side paper of hand written notes. Calculators are allowed, but probably won’t be useful. Use of cellphones, laptops, etc is not allowed.

• The problems are not necessarily in order of difficulty. Do the problems in whichever order you find most natural.

• A correct answer does not guarantee full credit and a wrong answer does not guarantee loss of credit. You should concisely indicate your reasoning and show all relevant work. The grade on each problem is based on our judgment of your level of understanding as reflected by what you have written.

• If we can’t read it, we can’t grade it.

• If you don’t understand a problem, please ask.
Problem 1 (25 Points)
Prove or disprove the following statements. Your answer will only receive points if your arguments are correct.

(a) A source code $C$ is uniquely decodable if and only if it is prefix-free.

(b) Let $\{u(t) : \mathbb{R} \to \mathbb{C}\}$ be a $L_2$ and band-limited function to $W$. Then $u(t)$ can be specified by its samples spaced at $T = 1/2W$, for each $t \in \mathbb{R}$.

(c) Assume any vector $v \in \mathcal{V}$ can be written as a linear combination of vectors $\{v_1, v_2, ..., v_n : v_i \in \mathcal{V}\}$. If this set of vectors spans $\mathcal{V}$, then the dimension of the vector space is $n$.

(d) The Fourier transform of the function: $u(t) = \text{sinc}(2Wt - k)e^{2\pi i \delta(t - \frac{k}{2W})}$, is: $\hat{u}(f) = (1/2W)e^{-2\pi i (f - \delta)k/2W}\text{rect}(\frac{f}{2W})$.

(e) Two functions $u(t)$ and $v(t)$, where $\{u(t), v(t) : \mathbb{R} \to \mathbb{R}\}$, are called $L_2$-equivalent if the difference between their Fourier transforms has zero energy.
Problem 2 (10 points)
Assume we can take the high-rate assumption for $f \sim \mathcal{N}(0, \sigma^2)$. Consider a high-rate scalar quantizer with spacing $\Delta$.

(a) Relate $\sigma^2$ to the MSE.

(b) Relate $\sigma^2$ to $\bar{L}$.

(c) We are offered to transmit a sequence of Gaussian random variables every $T$ seconds. For a given MSE, explain how we can trade-off energy $\sigma^2$ with $T$. 


Problem 3 (30 points)

Consider a discrete memoryless source with a source alphabet $\mathcal{A} = \{a_1, a_2, a_3, \ldots, a_{n-1}, a_n\}$. The probability of each symbol is: $\{\frac{1}{3}, \left(\frac{1}{3}\right)^2, \left(\frac{1}{3}\right)^3, \ldots, \left(\frac{1}{3}\right)^{n-1}, 1 - \sum_{k=1}^{n-1} \left(\frac{1}{3}\right)^k\}$.

(a) Use Huffman’s algorithm to encode the source output in bits.

(b) Based on (a), calculate the average codeword length for Huffman’s encoding.

(c) Using a large block code to encode the source, which is the minimum average codeword length that can be achieved?

(d) Compare the results you got in (b) and (c). Which approach is better in terms of average codeword length? Explain briefly.

(You can use the following result without proof: $\sum_{k=1}^{n-1} \frac{k}{3^k} < \frac{3^{n-2} - 1}{2(\log 3 - 1)3^{n-2}}$.)
**Problem 4 (35 points)**

Assume \( \hat{u}(f) \) is the Fourier Transform of a continuous function \( u(t) \), as shown in the next figure:

![Figure 1: \( \hat{u}(f) \)](image)

(a) Find the minimum sampling rate \( R \) (i.e. time intervals \( T = 1/R \)) that the function \( u(t) \) should be sampled at in order to avoid aliasing, given that a standard sync-reconstruction process is used:

\[
u(t) = \sum_{k=-\infty}^{k=+\infty} u(kT) \text{sinc}(\frac{t}{T} - k),
\]

and that you are not allowed to filter or perform any other operation on the function before the sampling process.

(b) If you are not restricted to the previous reconstruction method, can the function \( u(t) \) be sampled at a rate \((W_1 + W_2)\) (i.e. time intervals of \( T' = \frac{1}{W_1 + W_2} \)) and perfectly recovered by its samples? If not, which is the minimum rate \((R_{min})\) that the function can be sampled at? Express \( u(t) \) in that case as a function of its samples \( \{u(kT_{min}); k \in \mathbb{Z}\} \), where \( T_{min} = 1/R_{min} \).

(c) Consider a PAM baseband system and let \( \{v_k\} \) to be symbols for transmission. Assume that the modulator uses a filter \( p(t) \) and the generated waveform \( v(t) = \sum_k v_k p(t - kT_0) \) is transmitted. At the receiver, the demodulator uses a filter \( q(t) \) and the output of this filter is \( T_0\)-space sampled (assume there is no noise). If \( \hat{p}(f) \times \hat{q}(f) = \hat{u}(f) \), the function shown in Figure 1, is it possible that the symbols can be received without intersymbol interference? If yes, specify the values of \( \alpha, W_1 \) and \( W_2 \) in terms of \( T_0 \) so that intersymbol interference is always avoided.