Midterm

- You have 90 minutes to complete the quiz.

- This is an open-book quiz. You are allowed to have a double-side paper of hand written notes. Calculators are allowed, but probably won’t be useful. Use of communication devices is not allowed.

- The problems are not necessarily in order of difficulty. Do the problems in whichever order you find most natural.

- A correct answer does not guarantee full credit and a wrong answer does not guarantee loss of credit. You should concisely indicate your reasoning and show all relevant work. The grade on each problem is based on our judgment of your level of understanding as reflected by what you have written.

- If we can’t read it, we can’t grade it.

- If you don’t understand a problem, please ask.
Problem 1 (25 Points)
Prove or disprove the following statements. Your answer will only receive points if your arguments are correct.

(a) Any bounded $L_1$ function $\{u(t) : \mathbb{R} \to \mathbb{C}\}, |u(t)| < A$ is also $L_2$.
Answer: TRUE. Since $|u(t)| < A, |u(t)|^2 < A|u(t)|$. Therefore,
\[
\int_{-\infty}^{\infty} |u(t)|^2 dt < \int_{-\infty}^{\infty} A|u(t)| dt \\
= A \int_{-\infty}^{\infty} |u(t)| dt,
\]
which is finite since $u(t)$ is $L_1$. Hence, $u(t)$ is $L_2$.

(b) Consider a discrete memoryless source with $i$-th symbol probability $p_i$, $p_1 \geq p_2 \geq \ldots p_n$. The Huffman code $C$ corresponding to this source was constructed. Then it turned out, that two smallest symbol probabilities $p_{n-1}, p_n$ were reported with error, while the values of $p_1, p_2, \ldots p_{n-2}$ were correct. If the error is at most $p_n/2$, then $C$ is also the corresponding Huffman code for the correct set of symbol probabilities, and there is no need to construct a new code.
Answer: FALSE. After the correction the probability $p_{n-1}$ of symbol $n-1$ may become greater than $p_{n-2}$. This will change the two least probable symbols from $(n, n-1)$ to $(n, n-2)$. Then the symbols $n, n-1$ will not be paired together in the first iteration of the Huffman code construction, and the resulting code will be different from $C$.

(c) A binary sequence $S$ of $4k$ bits can also be regarded as a sequence of $k$ hexadecimal symbols. We compress $S$ by LZ77 two times independently: first time – as a sequence of bits; second time – as a sequence of hexadecimal digits. Then on any iteration the longest match (between the not-yet-encoded symbols and a string of symbols starting in the window) will have equal bit-length both times.
Answer: FALSE. If in the first case the match length is not divisible by 4 than in the second case the match string will be shorter. E.g.
010010100010
010010100011
matched length=11 bits
0100 1010 0010
0100 1010 0011
matched length=2 hex digits=8 bits

(d) Assume that continuous functions $f_1(t), f_2(t)$ are bandlimited to $[\Delta_1 - W, \Delta_1 + W], [\Delta_2 - W, \Delta_2 + W]$, respectively, $|\Delta_1 - \Delta_2| \leq 2W$. Then the sum $f_1(t) + f_2(t)$ can always be perfectly reconstructed from its samples with rate $\pi W$.
Answer: FALSE. The sampling rate $\pi W$ is enough only if $f_1(t) + f_2(t)$ have bandwidth at most $\pi W,$
which is not true if $|\Delta_1 - \Delta_2| > (\pi - 2)W$. (The functions were meant to be $L_2$, but that was not mentioned explicitly, so the answer “False, since $f_1(t) + f_2(t)$ is not $L_2$” was also considered correct.)

Problem 2 (10 points)
Show that if an $L_2$ function $u(t)$ is time-limited to $[\Delta - T/2, \Delta + T/2]$, then its Fourier transform $\hat{u}(f)$ can be perfectly reconstructed from a countable sequence of its samples $\hat{u}(f_k)$, by deriving the corresponding sampling equation from the Fourier series expansion of $u(t)$.

Answer:
The function $u(t)$ is $L_2$ and time-limited, thus, it is also $L_1$. The shifted Fourier series over $[\Delta - T/2, \Delta + T/2]$ for $u(t)$:

$$u(t) = \text{l.i.m.} \sum_k \hat{u}_k e^{2\pi ikt} \text{rect}(\frac{t - \Delta}{T}).$$

Taking the Fourier transform, we get a continuous function

$$\hat{u}(f) = \sum_k \hat{u}_k T \text{sinc}(T(f - k/T)) e^{-2\pi i\Delta(f - k/T)}.$$ 

Samples at $f = k/T$:

$$\hat{u}(k/T) = \sum_{k'} \hat{u}_{k'} T \text{sinc}(T(k/T - k'/T)) e^{-2\pi i\Delta(k/T - k'/T)}) = \hat{u}_k T.$$ 

Hence,

$$\hat{u}_k = \frac{1}{T} \hat{u}(k/T).$$

Therefore,

$$\hat{u}(f) = \sum_k \hat{u}(k/T) \text{sinc}(T(f - k/T)) e^{-2\pi i\Delta(f - k/T)}.$$ 

Problem 3 (35 points)
Let $(X,Y)$ be a pair of random variables with joint pdf

$$f_{X,Y}(x,y) = \begin{cases} 
    1/2, & \text{if } |x| + |y| < 1 \\
    0, & \text{otherwise} 
\end{cases}.$$ 

(1)

(a) Let $(X,Y)$ be quantized with a single quantization region. Find the representation point that minimizes MSE, and the value of MSE.

Answer:
The optimal representation point $a_1$ is given by the mean of $(X,Y)$. Since the pdf is an even function, the mean is zero: $(a_1, a_1') = (0, 0)$. The MSE is

$$MSE = \int_{|x|+|y|<1} f_{X,Y}(x,y)(x^2 + y^2) dxdy = \int_{x+y<1} \int_{x:y>0} \frac{1}{2}(x^2 + y^2) dxdy$$

$$= 2 \int_{x=0}^{1} \int_{y=0}^{1-x} x^2 + y^2 dydx = 2 \int_{x=0}^{1} (1 - x)x^2 + \frac{(1-x)^3}{3} dx$$

$$= 2(1/3 - 1/4 + \frac{1}{3 \cdot 4}) = 1/3 \approx 0.33.$$ 

(2)
(b) Now \((X,Y)\) is quantized with 2 quantization regions (1-bit quantizer). The Lloyd-Max algorithm is started with \((a_1, a'_1) = (0, 0), (a_2, a'_2) = (1, 0)\). Explain where the Lloyd-Max algorithm converges for this starting point. Find the resulting \((a_1, a'_1), (a_2, a'_2)\).

**Answer:**

The representation points will stay on the horizontal line \(y = 0\), and the boundary between the quantization regions will be vertical. The representation points will drift to the left, until \(a_1 = -a_2\) and the boundary becomes \(x = 0\). The final quantization regions are the regions with \(x < 0\) and \(x > 0\), respectively. The regions are triangles. The conditional pdf in each region is \(f'_{X,Y}(x,y) = f_{X,Y}(x,y)/\Pr[x > 0] = 1\) (when \((x,y)\) lies in the corresponding region).

\[
a_2 = -a_1 = \mathbb{E}[x|x > 0] = \int_{x>0} x f'_{X,Y}(x,y) \, dx \, dy = \int_{x=0}^{1} \int_{y=x-1}^{1-x} x \, dy \, dx
\]

\[
= \int_{x=0}^{1} 2(1-x) \, dx = 1 - 2/3 = 1/3
\]

\(f'_{X,Y}(x,y)\) is symmetric with respect to \(y = 0\), hence \(a'_1 = \mathbb{E}[y|x < 0] = 0, a'_2 = \mathbb{E}[y|x > 0] = 0\). Finally, \((a_1, a'_1) = (-1/3, 0), (a_2, a'_2) = (1/3, 0)\).

(c) Find the MSE for the quantizer that you have found in the previous part.

**Answer:**

Since both the quantization regions, and the representation points are symmetric with
respect to $x = 0$, MSE is equal in both regions and, thus, equal to the overall MSE.

$$MSE = MSE_{x>0} = \int_{x>0} [(x - a_2)^2 + (y - a'_2)^2]f_{X,Y}(x,y)\,dx\,dy$$

$$= \int_{x=0}^{1} \int_{y=x-1}^{1-x} (x - 1/3)^2 + y^2\,dy\,dx$$

$$= \int_{x=0}^{1} 2(1-x)(x-1/3)^2 + \frac{2(1-x)^3}{3}\,dx$$

$$= 2[-1/4 + (1/3 + 2/9) - (2/6 + 1/18) + 1/9 + \frac{1}{3\times4}]$$

$$= 2/9 \approx 0.22. \tag{4}$$

(d) Now the Lloyd-Max algorithm is started with $(a_1, a'_1) = (0,0), (a_2, a'_2) = (1/2, 1/2)$. Explain where the Lloyd-Max algorithm converges for this starting point. Find the resulting $(a_1, a'_1), (a_2, a'_2)$.

**Answer:**
The representation points will stay on the line $x = y$, and the boundary between the quantization regions will have the form $y = c - x$. The representation points will drift down along the line $x = y$, until $a_1 = -a_2$ and the boundary becomes $y = -x$. The final quantization regions are the regions with $x + y < 0$ and $x + y > 0$, respectively. The regions have rectangular form. Since $(X,Y)$ is distributed uniformly, its conditional mean in each region is the center of the corresponding rectangle. Hence, $(a_2, a'_2) = -(a_1, a'_1) = (1/4, 1/4)$.

![Diagrams](image)

(e) Find the MSE for the quantizer that you have found in the previous part, and compare it with MSE for the quantizers considered before.

**Answer:**
The conditional pdf in each quantization region is $f'_{X,Y}(x,y) = f_{X,Y}(x,y) / \Pr[x+y>0] = 1$ (when $(x,y)$ lies in the corresponding region). Since both the quantization regions, and the representation points are symmetric with respect to $x + y = 0$, MSE is equal in both regions and, thus, equal to the overall MSE. In the MSE calculation the integral is taken over a
$$\sqrt{2}/2 \times \sqrt{2} \text{ rectangle:}$$

$$MSE = MSE_{x+y>0} = \int_{x+y>0} [(x - a_2)^2 + (y - a'_2)^2] f'_{X,Y}(x, y) dxdy$$

$$= \int_{x+y>0} [(x - 1/4)^2 + (y - 1/4)^2] f'_{X,Y}(x, y) dxdy$$

$$= \int_{s=-\sqrt{2}/4}^{\sqrt{2}/4} \int_{t=-\sqrt{2}/2}^{\sqrt{2}/2} s^2 + t^2 dt ds$$

$$= \int_{s=-\sqrt{2}/4}^{\sqrt{2}/4} \sqrt{2} s^2 + \frac{2(\sqrt{2}/2)^3}{3} ds$$

$$= \sqrt{2} \times \frac{2(\sqrt{2}/4)^3}{3} + \frac{\sqrt{2}}{2} \times \frac{2(\sqrt{2}/2)^3}{3}$$

$$= 5/24 \approx 0.208.$$

This quantizer gives better MSE than 1-point quantizer, and the quantizer from part (b), although that quantizer is also found by the Lloyd-Max algorithm.

(f) More quantization regions can be used in a similar manner. Suppose that we do not use discrete encoding after quantization, and our 2D quantizer is combined with the modulator (channel encoder). What is the simplest way to construct the signal constellation for this scenario?

Answer:

The most straightforward way is to regard the sample space of 2D random variable $(X, Y)$ as the signal space (complex plane). In other words, the real and the imaginary part of signals are taken directly from the quantizer: quantizer representation points $\{(a_i, a'_i)\}_{i=1,2,\ldots}$ ⇒ signal constellation $\{a_i + ia'_i\}_{i=1,2,\ldots}$.

(g) Which quantizer, considered in this problem, should be used for the modulation scheme from part (f), if maximal signal power $P = 1/9$ is allowed?

Answer:

For the quantizer from part (b) $a_2^2 + a'_2 = 1/9 \leq P$. For the quantizer from part (d) $a_2^2 + a'^2 = 1/8$, which exceeds the power constraint 1/9. Therefore, the quantizer from part (b) should be used.

Problem 4 (30 points)

In this problem we use $\mathcal{F}[:], \mathcal{F}^{-1}[:]$ to denote the operations of taking the Fourier transform and the inverse Fourier transform, respectively.

(a) For $A \geq B > 0$ calculate $\int_{-\infty}^{\infty} \text{sinc}(A t) \text{sinc}(B(\tau - t)) dt$.

Answer:

Convolution in the time domain corresponds to multiplication in the frequency domain.
Therefore,
\[
\int_{-\infty}^{\infty} \text{sinc}(At)\text{sinc}(B(\tau - t))dt = \text{sinc}(At) * \text{sinc}(Bt)
\]
\[
= \mathcal{F}^{-1}\left[ \frac{\text{rect}(f/A)}{A} \frac{\text{rect}(f/B)}{B} \right]
\]
\[
= \mathcal{F}^{-1}\left[ \frac{1}{A} \frac{\text{rect}(f/B)}{B} \right]
\]
\[
= \frac{1}{A} \mathcal{F}^{-1}\left[ \text{rect}(f/B) \right]
\]
\[
= \frac{1}{A} \text{sinc}(B\tau).
\]

Let \( u(t) \) be a continuous bounded function. For any positive \( A, B \) let
\[
\hat{u}_A(f) = \mathcal{F}[u(t)\text{rect}(t/2A)]
\]
\[
u_{A,B}(t) = \mathcal{F}^{-1}[\hat{u}_A(f)\text{rect}(f/2B)].
\]  

(b) Show that for any \( A > 0 \)
\[
\lim_{B \to \infty} \int_{-\infty}^{\infty} |\nu_{A,B}(t) - u(t)\text{rect}(t/2A)|^2dt = 0.
\]  

Answer:
Function \( u(t)\text{rect}(t/2A) \) is bounded and time-limited, and thus \( \mathcal{L}_2 \). Hence, \( \hat{u}_A(f) = \mathcal{F}[u(t)\text{rect}(t/2A)] \) is also \( \mathcal{L}_2 \), and the limit follows from the Plancherel II theorem (inverse transform for \( \hat{u}_A(f) \)).

(c) Does the limit
\[
\lim_{A=B \to \infty} \int_{-\infty}^{\infty} |\nu_{A,B}(t) - u(t)|^2dt
\]  
exist? If it does, find its value.

Answer:
Consider \( u(t) = 1 \). Then \( \nu_{A,B}(t) = 2B\text{rect}(t/2A) * \text{sinc}(2Bt) \) decays quickly for \( |t| > A \). In particular, for \( \Delta > 0 \)
\[
u_{A,B}(A + \Delta) = 2B \int_{-\infty}^{\infty} \text{sinc}(2B\tau)\text{rect}((A + \Delta - \tau)/2A)d\tau
\]
\[
= 2B \int_{\Delta}^{\Delta+2A} \text{sinc}(2B\tau)d\tau
\]
\[
= 2B \int_{\Delta}^{\Delta+2A} \frac{\sin(2B\pi\tau)}{2B\pi\tau}d\tau
\]
\[
\leq \int_{\Delta}^{\Delta+2A} \frac{1}{\pi\tau}d\tau
\]
\[
= \frac{\ln(\Delta + 2A) - \ln(\Delta)}{\pi}
\]
\[
= \frac{\ln(1 + 2A/\Delta)}{\pi}.
\]
Hence, for $\Delta \geq 2A/(e^{\pi/2} - 1)$ we have $u_{A,B}(A + \Delta) \leq 1/2$, and

$$
\int_{-\infty}^{\infty} |u_{A,B}(t) - u(t)|^2 dt \geq \int_{|t| > A + \Delta} |u_{A,B}(t) - 1|^2 dt \\
\geq \int_{|t| > A + \Delta} |1/2 - 1|^2 dt \\
\geq \int_{|t| > A + \Delta} 1/4 dt = \infty.
$$

Therefore, the integral is infinite for any $A, B$, and limit (8) does not exist.