Final Exam

- You have 180 minutes to complete the exam.
- This is a open-book quiz. Calculators are allowed, but probably won’t be useful.
- The problems are not necessarily in order of difficulty. Do the problems in whichever order you find most natural.
- A correct answer does not guarantee full credit and a wrong answer does not guarantee loss of credit. You should concisely indicate your reasoning and show all relevant work. The grade on each problem is based on our judgment of your level of understanding as reflected by what you have written.
- If we can’t read it, we can’t grade it.
- If you don’t understand a problem, please ask.
Problem 1 (24 points/6 points per item)

Prove or disprove the following statements. Your answer will only receive points if your arguments are correct. If you give the wrong answer but present reasonable arguments, you may receive partial credit.

(a) An MIT student is walking down the infinite corridor, talking on his cell phone with his friend about his success on the 6.450 final. When he arrives at Lobby 10, he takes a turn and decides to continue the conversation outdoors, enjoying the warm sunshine at the open field at Killian court. As a consequence of his decision, the coherence frequency of his cell phone connection will decrease.

(b) There is no loss when performing ML detection on a vector of received symbols when compared to performing ML detection on a symbol-by-symbol basis.

(c) A discrete source can be compressed arbitrarily close to its entropy in bits only when the probability of each symbol is a power of 2.

(d) Given a certain rake receiver, it will have a better performance with a system at 2.4 GHz than with a system with the same bandwidth at 800 MHz.

(e) The random process \( Z(t) = (t^{1/2} - a)^2 X \), where \( X \sim N(0, \sigma^2) \), \( a \in \mathbb{R} \), is a Gaussian random process.

(f) For a baseband function, reducing the bandwidth of the system reduces the number of taps required for a discrete-time model of the channel.
Problem 2 (20 points)

In the first two items of this problem we will analyze an alternative method for converting signals from baseband to passband. Consider the following modulation scheme:

\[ \cos(2\pi f_c t) \]
\[ u(t) \]
\[ g_i(t) \]
\[ x(t) \]

Figure 1: Baseband to passband modulation.

where

\[ g_i(t) = \frac{1}{T} \text{sinc} \left( \frac{t}{T} \right) \cos(2\pi t(f_c + mW)), \quad (1) \]

\[ m = -1, 0 \text{ or } 1, \quad f_c \gg 2W \text{ and } T = 1/2W. \]

(a) (6 pts) Analyze the output \( x(t) \) of this system when \( m = 1 \) and \( u(t) \) is a real baseband function limited to \([-W, W]\). Compare the case when \( m = 1 \) to when \( m = 0 \), and discuss the implementation issues of this scheme.

(b) (4 pts) How can the systems with \( m = 1 \) and \( m = -1 \) be combined in parallel in order to create a QAM signal? Compare this with at least one we have seen.

On the last two items of this problem, we will investigate the effect of aliasing on a signal with infinite bandwidth.

(c) (5 pts) Consider a function \( f(t) \) given by

\[ f(t) = \sum_{k=1}^{m} \frac{k^{-\alpha}}{T} \text{sinc} \left( \frac{t}{T} \right) \cos \left( \frac{2\pi t(k - 1)}{T} \right) \quad (2) \]

For a fixed \( m \), can \( f(t) \) be scaled into an ideal Nyquist pulse with interval \( T \)? For which real values of \( \alpha \) does this hold?

(d) (5 pt) Now consider \( m \to \infty \). For which values of \( \alpha \) does the sampling approximation \( \sum_k f(kA)\text{sinc}(t/A - k) \) converge to a bounded \( L_2 \) function for any \( A > 0 \)?
Problem 3 (19 points)

In this problem we will investigate the use of ternary signal sets with unequal probability distribution over an AWGN channel. We will also look at the effect of adding a signal with energy 0 in the signal set.

(a) (4 pts) Given a ternary source $S$, (i.e., a source with an alphabet of three symbols), we wish to find a probability distribution over the source symbols in order to achieve a target entropy such that $H(S) = 1.5$ bits. Find a probability mass function that achieves this entropy. [Hint: consider two equally likely symbols.]

(b) (6 pts) Consider now that the symbols of $S$, with the pmf of part (a), are mapped onto signals of a 3-PAM modulation scheme, with signal set $x_1 = 0$, $x_2 = a$ and $x_3 = -a$. If you haven’t found a distribution that satisfies (a), just suggest some distribution of the form $(p, p, 1 - 2p)$, $0 < p < 1$. Use a mapping that minimizes the average energy of the signal set. Considering that $x_i$ is transmitted, after demodulation the received signal is $Y = x_i + Z$, where $Z \sim N(0, N_0/2)$. Suggest an appropriate MAP or ML scheme for this method, and calculate the average symbol error probability for the scheme you have developed. Be sure to present all your steps.

(c) (5 pts) Now assume that the symbols of $S$ are mapped onto an “orthogonal” code, where $x_1 = (0, a)$, $x_2 = (0, 0)$ and $x_2 = (a, 0)$, again using a mapping that minimizes the average energy. The corresponding received signal, given that $x_i$ is transmitted, is $Y = x_i + Z$ where each component of $Z$ has a distribution $N(0, N_0/2)$. Once again, suggest an appropriate MAP or ML scheme for this method, and calculate the average symbol error probability for the scheme you have developed. Be sure to present all your steps.

(d) (4 pts) Compare the ternary schemes presented in items (a) and (b) with their binary counterparts (i.e., antipodal and orthogonal signal sets). Consider that the binary schemes have independent and equally likely signal points. What are the advantages/disadvantages of the suggested schemes?
Problem 4 (19 points)
Consider a digital modulation scheme where a pulse $u_i(t)$, $i = 1, 2, 3, \text{ or } 4$, is transmitted over an AWGN channel. The pulse $u_i(t)$ is given by:

$$u_i(t) = \begin{cases} A\sqrt{2f_0} \cos (2\pi f_0 t + \phi_i) & \text{if } 0 \leq t \leq 1/f_0 \\ 0 & \text{otherwise} \end{cases}$$ (3)

where $A > 0$ and $\phi_1 = 0$, $\phi_2 = \pi/2$, $\phi_3 = \pi$ and $\phi_4 = -\pi/2$.

(a) (5 pts) Draw the 2-dimensional signal space representation of this signal. Present a block diagram of the optimal detector when the observed signal is the sum of the transmitted signal and a white Gaussian process $Z(t)$ with spectral density $N_0/2$. Denote the coordinate pair corresponding to signal $u_i$ as $u_i = (x_i, y_i)$ and to the additive white Gaussian noise as $z = (z_1, z_2)$.

(b) (5 pts) Considering that all signals are equally likely, illustrate the decision boundaries on the signal space diagram of this scheme and calculate the error probability (or give your best estimate) when maximum likelihood detection is used.

(c) (5 pts) Now consider that your front end receiver is subject to correlated noise on each dimension, such that the detected signal is of the form $y = u_i + z + a$, where $a = (a, a)$ and $a \sim N(0,1)$. Give an expression for the probability density function of $y$ given that $u_i$ was transmitted.

(d) (4 pts) Give an expression or your best approximation for the new probability of symbol error when ML detection is used. How does the correlated noise affect the decoding procedure and the overall error probability? [Hint: Consider an appropriate orthogonal transformation of the noise components. A figure may help.]
Problem 5 (18 points)

In this problem we will try to create a basic tap model for a wireless channel, and understand the performance of a simple precoding scheme for the developed model. Consider a wireless channel with two main paths, where the linear time variant impulse response can be modeled as

\[
\hat{G}(f, t) = \sum_{j=1}^{2} \gamma_j(t) \exp\{-2\pi if \tau_j(t)\} \tag{4}
\]

(a) (6 pts) A PAM transmission scheme with period \(T\) is used over the channel \(\hat{G}(f, t)\), where the transmitted signal is \(u_1(t) = \sum_k u_{1,k} \text{rect}(t/T - k)\). The output of the channel, denoted by \(y_1(t)\), is sampled at instants \(t = kT\) to produce the output sequence \(y_{1,k} = y_1(kT)\). Considering that \(T \ll T_{\text{coh}}\), we approximate \(\tau_1(t) \approx T/4, \tau_2(t) \approx 6T/5, \gamma_1(t) \approx \gamma_1\) and \(\gamma_2(t) \approx \gamma_2\). Create a two-tap model for the channel, and give an expression for the tap gains \(g_0\) and \(g_1\) in terms of \(\gamma_1\) and \(\gamma_2\). Express the output \(y_{1,k}\) in terms of the input \(u_{1,t}\). Justify your steps.

(b) (7 pt) Now assume that the transmitter has two transmission antennas, while the receiver has one receive antenna. The input signals for antenna 1 and 2 at instants \(t = kT\) are \(u_{1,k}\) and \(u_{2,k}\), respectively. The channel seen by the first transmit antenna and the receive antenna has the same two tap model you derived in item (a), and output \(y_{1,k}\). The channel of the second antenna can also be represented by a two tap model, but with channel gains \(h_0\) and \(h_1\), and output \(y_{2,k}\). The sampled output at the receiver antenna is the sum of the two channels, given by \(y_k = y_{1,k} + y_{2,k}\).

Assume that we wish to send the signals \(x_k\) through this two-input one-output scheme. The following channel precoding scheme is used at the receiver, where \(D\) represents a delay:

![Figure 2: Channel precoding scheme.](image)

The values \(a_k\) and \(b_k\) are pre-transmission gains, and can be freely adjusted by the transmitter. Give an expression for the output of this two-input-single-output scheme in terms of the precoding gains and the tap coefficients. Assuming that the transmitter has perfect knowledge of the tap gains, give the values of \(a_k\) and \(b_k\) at each transmission instant in order to eliminate inter-symbol interference and fading effects. [Hint: the output can be a delayed version of the input.]

(c) (5 pts) The channel taps have a Rayleigh distribution for each transmission time given by \(f_{g_1}(|g|) = f_{h_1}(|g|) = |g| \exp(-|g|^2/2)\). What is the average transmission energy required for the scheme suggested in the previous item?