Final Exam

- You have 180 minutes to complete the exam.
- This is a open-book quiz. Calculators are allowed, but probably won’t be useful.
- The problems are not necessarily in order of difficulty. Do the problems in whichever order you find most natural.
- A correct answer does not guarantee full credit and a wrong answer does not guarantee loss of credit. You should concisely indicate your reasoning and show all relevant work. The grade on each problem is based on our judgment of your level of understanding as reflected by what you have written.
- If we can’t read it, we can’t grade it.
- If you don’t understand a problem, please ask.
Problem 1 (16 points)
Prove or disprove the following statements. Your answer will only receive points if your arguments are correct. If you give the wrong answer but present reasonable arguments, you may receive partial credit.

(a) The performance of a wireless communication system, designed by company XYZ and employed in a mobile environment, suffers from fading. A graduate student from MIT, interning in company XYZ, proposes to use a higher carrier (i.e., two times the existing one) as a potential solution. Do you think it will help increase the performance of the system?
   It won’t help since the coherence time will be decreased (Doppler shifts are linear with the carrier frequency) and channel measurements will be outdated faster.

(b) Assume \( \{Z(t); t \in \mathbb{R}\} \) is a stochastic process. If \( Z(t) = \sum_k Z_k \text{rect}(t - k) \) and \( Z_k \sim \mathcal{N}(0, 1) \), then \( Z(t) \) is a Gaussian random process.
   FALSE. We do not know if \( Z_k \)’s are independent, so \( Z(t_1), Z(t_2), ..., Z(t_n) \) are not jointly Gaussian and \( Z(t) \) is not a Gaussian process.

(c) A stationary process \( \{Z(t); t \in \mathbb{R}\} \) can have finite energy.
   FALSE. Stationary processes have infinite energy.

(d) The inverse Fourier transform of \( \hat{u}(f) = e^{-|f^2+f|} \) is continuous and \( L_1 \).
   TRUE. \( \hat{u}(f) \) is \( L_1 \) and, therefore, its inverse is continuous. The fact that the inverse Fourier transform is also \( L_1 \) can be checked by noting that it is non-negative everywhere and using (4.43) from the book.
Problem 2 (14 points)

Consider a communication system operating over a wireless channel with two paths at carrier frequency $f_c$. The gain of both paths is constant equal to $\beta$ and their delays $\tau_i$ ($i = 1, 2$) are varying over time according to the equation: $\tau_i = \tau'_i \times t$, where $\tau'_i$ are random variables with uniform distribution: $\tau'_1 \sim U(0, \epsilon)$ and $\tau'_2 \sim U(\epsilon, 2\epsilon)$.

(a) [4 points] Find the system function $\hat{h}(f, t)$ of the channel.

Solution outline: $\hat{h}(f, t) = \beta(e^{-2\pi i f \tau'_1 t} + e^{-2\pi i f \tau'_2 t})$

(b) [5 points] Find the coherence time of the channel and give an intuitive explanation of what does it represent.

Solution outline: $T_{coh} = 1/2D$, where $D$ is the Doppler spread. Since $\tau'_i$'s are uniform random variables, the distribution of $D$ would be the following:

(c) [5 points] If the transmitted waveform has a bandwidth of $W$ and the coherence frequency of the channel is $W/10$, specify the minimum required sampling rate at the receiver to recover it.

Solution outline: Since the bandwidth of the signal is greater than the coherence bandwidth, or equivalently, the delay spread of the channel is greater than the symbol time, we will have frequency selective fading and so, the delayed versions of each symbol will be added constructively or distractively to each other causing ISI. In this case, the transmitted waveform can not be perfectly recovered by the samples of the received waveform (oversampling does not help, it needs to be equalized).
**Problem 3 (25 points)**

In this problem we will examine the performance of a receiver in the presence of non-AWGN noise.

Consider the following receiver which uses a matched filter followed by a sampler, whose output is fed to the detection block, as shown in Figure 1.

![Figure 1: Block diagram of the receiver.](image)

(a) [6 points] Assume that the input of the detection block is \( V = \pm a + Z \), where \( +a \) and \( -a \) are equiprobable and \( Z \) is a noise random variable with the following probability density function:

\[
    f_Z(z) = \frac{\lambda}{2} e^{-\lambda|z|}.
\]

Calculate the log likelihood ratio \( \text{LLR}(V) \) and write the MAP decision rule.

Solution outline: Similar to the case with AWGN noise described in section 8.3, the decision rule is: \( \hat{u} = a \) when \( v \geq 0 \), and \( \hat{u} = -a \) when \( v < 0 \)

(b) [5 points] Calculate the average probability of error \( Pr(e) \) and express it in terms of the received SNR.

Solution outline: Because of the symmetry:

\[
    Pr(e) = \int_0^\infty \frac{\lambda}{2} e^{-\lambda(v+a)} dv = \frac{e^{-\lambda a}}{2}
\]

The noise variance is:

\[
    \sigma_Z^2 = \int_{-\infty}^{+\infty} \frac{\lambda}{2} e^{-\lambda|z|} z^2 dz = \frac{2}{\lambda^2},
\]

so the SNR is:

\[
    SNR = \frac{a^2 \lambda^2}{2}
\]

and the error probability:

\[
    Pr(e) = \frac{e^{-\sqrt{2SNR} \lambda a}}{2}
\]

(c) [5 points] Compare your result in (b) with the average error probability in the case of Gaussian noise distribution. Find the required SNR in both cases for a typical \( Pr(e) \) of modern indoors wireless communication systems (feel free to choose a \( Pr(e) \) based on your intuition) and comment on it.

Solution outline: Assuming a packet length in the order of \( 10^4 \) bits length and an average packet error probability of \( 10^{-2} \), a typical \( Pr(e) \) is in the order of \( 10^{-6} \). Based on (b), the required SNR for this error probability is 86.1 or 19.3dB. For the Gaussian noise distribution we know that \( Pr(e) = Q(\sqrt{SNR}) \), so the required SNR is approximately 22.6 or 13.6dB. Thus, the required SNR for the Gaussian noise is 5.6dB less; this was expected since \( f_Z(z) \) decays slower than the gaussian and has larger tail.
(d) [9 points] We know that the filter matched to the transmission pulse maximizes the SNR at its output and the sampled value of its output at \( t = kT \) is a sufficient statistic to make the hypothesis testing. However, this is true only if the noise is additive white (flat spectral density). In systems where additive colored noise (not white) is inserted during the transmission, an extra filter can be used to make the noise at the input of the matched filter behave as white noise, as shown in the next figure:

![Block diagram of the receiver with a whitening filter.](image)

Figure 2: Block diagram of the receiver with a whitening filter.

Find the frequency response \( W(f) \) of the whitening filter if the covariance of the colored noise is \( K_Z(t) = \text{sinc}^2\left(\frac{t}{2}\right) \).

Solution outline: Using Fourier transform properties and \( K_Z(t) \), the power spectral density of the colored noise \( (S_Z(f)) \) can be calculated. The frequency response of the whitening filter is given by: \( S_{\text{white}}(f) = S_Z(f)|W(f)|^2 \).
Problem 4 (25 points)

In this problem we will examine hypothesis testing under different wireless channel settings.

(a) [8 points] Consider binary hypothesis testing between $H_0$ and $H_1$. The first hypothesis is represented by sending $n$ 0s and the second by sending $n$ identical symbols, each with energy $\frac{E}{n}$ in a system with a channel of two known non-zero taps of equal amplitude 1 and phase 0, and iid additive Gaussian noise of zero mean and energy 1.

Express the probability of error. Comment on its dependence, if any, on $n$.

How does this probability of error evolve if we have the number $m$, of equal taps, grow?

Solution outline: Here we have non-andipodal hypothesis testing. The probability of error does not depend on $n$, since the total energy transmitted remains the same. Increasing the number of paths increases the diversity of the system and so, the error probability is reduced, without increasing the SNR at the transmitter.

(b) [7 points] There is now some mismatch between the channel and its model. The first tap has amplitude $1 + \epsilon$ and the second tap has amplitude $1 - \epsilon$, where $\epsilon$ is unknown but small.

Express the probability of error and comment on its relation to our results in the previous question. What happens when $E$ grows or becomes very small?

Solution outline: Here the knowledge about the tap is not perfect. When $E$ is low, channel noise is the dominant source of errors while, when $E$ is high, the partial knowledge of taps amplitude increases error probability.

(c) [10 points] Consider the case of the first question, where we now have a single tap of amplitude 1 and phase 0, but two receive antennas, with mutually independent noise.

Express the probability of error and comment on how it compares to the case of the two-tap channel in the first question.

How does the probability evolve if we have the number $k$ of antennas with mutually independent noises grow? Compare the effect of the growth in $m$ and in $k$.

Solution outline: The error probability decreases exponentially with the number of antennas in the low SNR regime and linearly with the number of taps.
Problem 5 (20 points)

In this problem we will explore the use of signal representation in compression.

Consider a signal created in the following way:
\[ F(t) = \sum_{k=1}^{n} A_k 2^{-k} e^{i\pi k t} \]

where the \( A_k \)'s are uniform iid Bernoulli taking on values +1 and −1.

(a) [6 points] Provide a means of describing the above signal that intelligently takes into account its structure.

Solution outline: Taking the Fourier transform of \( F(t) \), leads to a very compact representation. It contains frequency components from \( 1/n \) to \( 1/2 \) and their amplitude depends on the frequency while their sign is a Bernoulli rv.

(b) [7 points] Suppose you are given only \( m < n \) bits. Explain how the error in reconstructing \( F \) evolves with \( m \) using your scheme for the previous question.

Solution outline: Since the more energy is contained in the higher frequency components, if \( m < n \) then it makes sense to keep only the \( m \) highest frequency components. The error in the reconstruction goes down exponentially with \( m \).

(c) [7 points] Discuss how the answer to the first two questions differ from the results that would be obtained from straightforward sampling followed by quantization.

Solution outline: Since the signal \( F(t) \) is bandlimited, it can represented by its samples, according to sampling theorem. However, infinite number of samples are required to described it, compared to the \( n \) in the frequency domain. If \( m < n \) bits are given (and a time-limited duration is examined), then the reconstruction error is reduced logarithmically with \( m \).