1 Overview

In the last lecture, we described algorithms for computing Nash equilibria in normal form games, analyzing their running time. Today, we revisit the Lemke-Howson algorithm [1], providing an example execution of the algorithm in a simple game. We then revisit the structure of the algorithm’s proof of correctness. Finally, we use this structure to show that non-degenerate 2-player games (defined later) have an odd number of Nash equilibria.

2 An Example Execution of the Lemke-Howson Algorithm

Let’s consider the symmetric 2-player game \((R, C = R^T)\), where the payoff matrix \(R\) of the row player is the following:

\[
R = \begin{pmatrix}
0 & 3 & 0 \\
2 & 2 & 2 \\
3 & 0 & 0 
\end{pmatrix}
\]

Following the description of the Lemke-Howson algorithm from last lecture, we consider the polytope \(R \cdot z \leq (1, 1, 1)^T\) and \(z \geq (0, 0, 0)^T\). The polytope with its 6 facets is shown in Figure 1. Each facet corresponds to making one of the constraints defining the polytope tight. Notice that every vertex of the polytope is the intersection of exactly three constraints. So the Lemke-Howson algorithm can be applied without need to perturb the polytope.

![Figure 1: Lemke-Howson polytope for the symmetric 2-player game defined by payoff matrix \(R\). \(R_i\) represents the \(i\)-th row of matrix \(R\).](image)

To find a Nash equilibrium of the game \((R, R^T)\), the algorithm performs a walk on the edges of the polytope, starting at vertex \((0, 0, 0)\) and proceeding a non-trivial democracy is reached. The concept
of a “democracy” was defined in the last lecture. For this specific game with 3 actions per player, a
democracy is a vertex \( z \) of the polytope where, for all \( i = 1, 2, 3 \), pure strategy \( i \) is represented at \( z \),
meaning that either \( z_i = 0 \) or \( R_i \cdot z = 1 \). A non-trivial democracy is a democracy different than \((0,0,0)\).

Following the algorithm’s description, we define \( v_0 = (0,0,0) \) as the starting point of the walk. Each of the 3 edges of the polytope adjacent to \( v_0 \) corresponds to un-tightening one of the constraints \( z_1 \geq 0 \),
\( z_2 \geq 0 \) or \( z_3 \geq 0 \), which are tight at \( v_0 \). Using strategy 3 as our special strategy, we un-tighten \( z_3 \geq 0 \),
while keeping the other two constraints tight. This corresponds to an edge \((v_0,v_1)\) of the polytope,
which lies on the \( z_3 \) axis, as shown in Figure 2. In the first step, the algorithm moves to vertex \( v_1 \).

![Figure 2: Lemke-Howson starts at \( v_0 = (0,0,0) \). In the first step, it un-tightens the constraint \( z_3 \geq 0 \) moving along the \( z_3 \) axis until it reaches vertex \( v_1 \). At \( v_1 \) strategy 2 is represented twice, strategy 1 once, and strategy 3 is not represented. This is signified by the label 12^2. At \( v_0 \) all strategies are represented, so the label of \( v_0 \) is 123.](image)

Clearly constraints \( z_1 \geq 0 \), \( z_2 \geq 0 \) are still tight at \( v_1 \) as they are tight for the entire edge \((v_0,v_1)\).
For \( v_1 \) to be a vertex of the polytope, there must be another constraint tight at \( v_1 \). If this constraint
were \( R_3 \cdot z \leq 1 \), then \( v_1 \) would have been a non-trivial democracy, and the algorithm would stop and
return \( v_1 \). However, the extra constraint that became tight at \( v_1 \) is \( R_2 \cdot z \leq 1 \). Hence, strategy 2 is
doubly represented at \( v_1 \), strategy 1 is singly represented at \( v_1 \), and strategy 3 is not represented at all.
This is why we associate the label 12^2 to vertex \( v_1 \) in Figure 2.

In a generic step of the algorithm, suppose that we are at some vertex \( v_i \) that is not a democracy. If strategy \( j \) is doubly represented at \( v_i \), we un-tighten either \( R_j \cdot z \leq 1 \) or \( z_j \geq 0 \), whichever defines an edge
\((v_i,v_{i+1})\) of the polytope that is different than \((v_{i-1},v_i)\). The walk goes to \( v_{i+1} \) and continues unless \( v_{i+1} \)
is a democracy. Figure 3 follows through the algorithm’s steps in our example. At \( v_1 \), where strategy 2
doubly represented, we un-tighten \( z_2 \geq 0 \), getting to vertex \( v_2 \). At \( v_2 \), strategy 1 is doubly represented,
so we proceed to un-tighten \( z_1 \geq 0 \), reaching \( v_3 \). \( v_3 \) is a democracy since \( R_1 \cdot z \leq 1 \), \( R_2 \cdot z \leq 1 \), and
\( z_3 \geq 0 \) are all tight. Thus, \((\frac{v_2}{v_1}, \frac{v_3}{v_2})\) corresponds to a Nash equilibrium of this game.

Notice that, had we chosen a different strategy, say strategy 1, as our special strategy for the first
pivoting step, our path to a democracy would have been different. The corresponding path is shown in
Figure 4. In our example, that path still leads to the same democracy, but this is not true in general.
3 Proof of Correctness of the Lemke-Howson Algorithm

3.1 Structure of the Proof

The proof of correctness of the Lemke-Howson algorithm was provided in the last lecture. Recall the structure of the proof. The two crucial components were the construction of a graph related to the Lemke-Howson polytope, and the use of a parity argument on that graph. The graph had the following vertices: (i) all democracies; and (ii) all vertices $v$ of the polytope satisfying property II from last lecture:
all of the strategies in \( \{1, \ldots, n - 1\} \) are represented at \( v \), exactly one of them is represented twice, and strategy \( n \) is not represented at all. Moreover, each vertex of Type (ii) had exactly two neighbors in the graph, while every democracy had exactly one neighbor. So the graph was a collection of simple paths and cycles, as in Figure 5. Then the parity argument was used as follows: given that the graph has

![Graph](image.png)

Figure 5: The structure of the graph constructed in the proof of correctness of the Lemke-Howson algorithm. Vertices of this graph correspond to vertices of the Lemke-Howson polytope that are either democracies or satisfy property Π. All vertices have degree \( \leq 2 \). Degree-1 vertices are exactly those corresponding to democracies. There is at least one degree-1 vertex, represented in blue, corresponding to vertex \((0, \ldots, 0)\) of the polytope.

at least one node of degree 1, namely the trivial democracy \((0, \ldots, 0)\), there must be another degree-1 node. This node must be a non-trivial democracy, i.e. a democracy different than \((0, \ldots, 0)\), and hence it corresponds to a Nash equilibrium of the game. In fact, since all endpoints of paths are democracies and only endpoints of paths are democracies, we learn from our construction that the number of non-trivial democracies is odd! In the next section, we will use this insight to prove that a non-degenerate game has an odd number of equilibria.

### 3.2 Comparison to Simplex

The Lemke-Howson algorithm bears some similarity to the Simplex algorithm for linear programming, as both algorithms perform pivoting steps on the vertices of a polytope. However, in the Simplex algorithm, there is a linear objective function that keeps improving, acting as a guide for the algorithm’s pivots. On the other hand, in the Lemke-Howson algorithm, there is no objective function that the algorithm’s pivots improve.
4 Odd Number of Equilibria

Using the structure of the graph constructed in the proof of correctness of the Lemke-Howson algorithm, we prove that there is an odd number of Nash equilibria in non-degenerate (not necessarily symmetric) 2-player games, defined next.

**Definition 1.** A 2-player game is called non-degenerate if and only if no mixed strategy of support size \( k \) has more than \( k \) pure best responses.

We show that any such game has an odd number of Nash equilibria. The intuition behind the proof is that the Nash equilibria of the game are in a one-to-one correspondence with the non-trivial democracies of an associated Lemke-Howson polytope, which we know are an odd number.

**Theorem 1.** A non-degenerate 2-player game \((R, C)\) has an odd number of Nash equilibria.

**Proof of Theorem 1:** The high level idea of the proof is this. First, using the symmetrization technique discussed in the previous lecture, we show that it suffices to prove the theorem for the number of symmetric Nash equilibria of symmetric games. We then establish a one-to-one correspondence between symmetric equilibria of symmetric games and non-trivial democracies of their associated Lemke-Howson polytopes.

We start with an easy observation.

**Lemma 1.** Without loss of generality, we can assume that \( R, C > 0 \).

**Proof of Lemma 1:** This is easy to see. By adding a large enough constant to all entries of \( R \) and \( C \), we define another game whose payoffs are all strictly positive and whose Nash equilibria can be shown to be in one-to-one correspondence with the Nash equilibria of \((R, C)\). \( \square \)

We argue next that it is enough to show that the symmetric game \( G_2 \) with the following payoff matrices in block form:

\[
\begin{pmatrix}
0 & 0 & R & C \\
C^T & R^T & 0 & 0
\end{pmatrix}
\]

has an odd number of symmetric Nash equilibria.

**Lemma 2.** Assume \( R, C > 0 \). There is a one-to-one correspondence between the Nash equilibria of \((R, C)\) and the symmetric Nash equilibria of \( G_2 \).

**Proof of Lemma 2:** We use \([x, y]\) to denote mixed strategies of the row player in \( G_2 \), where \( x \) is the component of the mixed strategy on the first block of rows:

\[
\begin{pmatrix}
0 & R, C
\end{pmatrix}
\]

and \( y \) the component of the mixed strategy on the second block of rows:

\[
\begin{pmatrix}
C^T, R^T & 0, 0
\end{pmatrix}
\]

Similarly, we denote mixed strategies of the column player of \( G_2 \) by \([x, y]\), where \( x \) and \( y \) are the components of the mixed strategy on the first and second blocks of columns respectively.

We define a mapping from the symmetric Nash equilibria of \( G_2 \) to the Nash equilibria of \((R, C)\). Suppose that \([x_1, y_1, x_2, y_2]\) is a symmetric Nash equilibrium of \( G_2 \). We map this equilibrium to the mixed strategy profile \((\hat{x} := \frac{x_1}{||x_1||}, \hat{y} := \frac{y_1}{||y_1||})\) in the game \((R, C)\), arguing that it is a Nash equilibrium. Suppose not. Without loss of generality, suppose that there is a mixed strategy \( \tilde{x} \) for the row player of the game \((R, C)\) such that \( \tilde{x}^T \hat{y} > \tilde{x}^T \hat{y} \). This implies that:

\[
|x_1| \cdot |y_1| \cdot \tilde{x}^T \hat{y} + y^T C^T x > |x_1| \cdot |y_1| \cdot \hat{x}^T \hat{y} + y^T C^T x
\]

i.e. that strategy \([|x_1| \tilde{x}, y]\) is strictly better than \([x, y]\) for the row player of \( G_2 \) against the strategy \([x, y]\) of the column player, contradicting that \(([x, y], [x, y])\) is a Nash equilibrium of \( G_2 \).
To complete the proof of the lemma, we have to argue that our mapping is onto and that no two symmetric Nash equilibria of \( \mathcal{G}_2 \) map to the same Nash equilibrium of \((R, C)\). Suppose that \((x, y)\) is a Nash equilibrium of \((R, C)\), where \(u = x^T R y\) and \(v = x^T C y\) are respectively the payoffs of the row and column players. It suffices to show that \( \left[ \frac{w}{u+v}, \frac{v}{u+v} \right] \) is a symmetric Nash equilibrium of \( \mathcal{G}_2 \), and that it is the unique symmetric Nash equilibrium of \( \mathcal{G}_2 \) of the form \( ([\alpha x, \beta y], [\alpha x, \beta y]) \), where \(\alpha, \beta \in \mathbb{R}_{\geq 0}\). Let us first show that the proposed symmetric strategy profile is a symmetric Nash equilibrium of \( \mathcal{G}_2 \). Suppose not. Then there exists some mixed strategy \([x', y']\) for the row player of \( \mathcal{G}_2 \) such that
\[
\frac{v}{u} x^T R y + \frac{u}{u+v} y^T C^T x > \frac{u \cdot v}{(u+v)^2} (x^T R y + y^T C^T x) = \frac{u \cdot v}{u+v};
\]
i.e.
\[
v \cdot x^T R y + u \cdot x^T C y' > u \cdot v. \quad (1)
\]
However, notice that \(x^T R y \leq |x'|_1 \cdot x^T R y = |x'|_1 \cdot u\) and \(x^T C y' < |y'|_1 \cdot x^T C y = |y'|_1 \cdot v\), since \((x, y)\) is a Nash equilibrium of \((R, C)\). Hence,
\[
v \cdot x^T R y + u \cdot x^T C y' < u \cdot v \cdot (|x'|_1 + |y'|_1) = u \cdot v,
\]
which contradicts (1).

To show uniqueness, suppose that \(([\alpha x, \beta y], [\alpha x, \beta y])\) is a symmetric Nash equilibrium of \( \mathcal{G}_2 \), for some \(\alpha, \beta \geq 0\). Clearly, \(\alpha + \beta = 1\). We distinguish the following cases:

- \(\alpha = 0, \beta = 1\): This is impossible as, if \(\alpha = 0\), then the row player of \( \mathcal{G}_2 \) would be able to strictly improve his expected payoff by making \(\alpha = 1, \beta = 0\), contradicting that \(([\alpha x, \beta y], [\alpha x, \beta y])\) is a Nash equilibrium of \( \mathcal{G}_2 \). (Recall that \(R, C > 0\) by assumption.)

- \(\alpha = 1, \beta = 0\): This is impossible for similar reasons.

- \(\alpha, \beta \neq 0\): For any pure strategy \(i\) in the support of \(x\), the expected payoff of the row player of \( \mathcal{G}_2\) if he was to switch to this strategy would be \(\beta e_i^T R y = \beta u\). For any pure strategy \(j\) in the support of \(y\), the expected payoff of the row player of \( \mathcal{G}_2\) if he was to switch to this strategy would be \(\alpha e_j^T C x = \alpha x^T C e_j = \alpha v\). For \(([\alpha x, \beta y], [\alpha x, \beta y])\) to be a Nash equilibrium of \( \mathcal{G}_2\), it must be that \(\beta u = \alpha v\). Given that \(\alpha + \beta = 1\), it follows that \(\alpha = \frac{u}{u+v}\) and \(\beta = \frac{v}{u+v}\).

\(\square\)

Next we show that the non-degeneracy of \((R, C)\) is inherited by \( \mathcal{G}_2 \).

**Lemma 3.** If \((R, C)\) is non-degenerate, then the symmetric game \( \mathcal{G}_2 \) as defined above is also non-degenerate.

**Proof:** Suppose that \([x, y]\) is a mixed strategy of the column player in \( \mathcal{G}_2\). We denote by \(k_x\) the size of the support of \(x\), and by \(k_y\) the size of the support of \(y\).

Among the row player’s strategies in the first block of \( \mathcal{G}_2\):
\[
\left( \begin{array}{cc} 0 & 0 \\ R, C \end{array} \right),
\]
at most \(k_y\) are best responses to the column player’s strategy, because of the non-degeneracy of \((R, C)\). Using the same argument, we can conclude that, among the row player’s strategies in the second block of \( \mathcal{G}_2\):
\[
\left( \begin{array}{cc} C^T & R^T \end{array} \right), 0, 0 \right),
\]
at most \(k_x\) strategies are best responses to the column player’s strategy. Therefore, the row player has at most \(k_x + k_y\) best responses to \([x, y]\). A symmetric argument holds for the column player, concluding the proof of the lemma. \(\square\)

Lemmas 1, 2 and 3 establish that, to conclude the proof of Theorem 1, it suffices to show that a non-degenerate symmetric game has an odd number of symmetric Nash equilibria. This is established by the following theorem.
Theorem 2. A non-degenerate symmetric game has an odd number of symmetric Nash equilibria.

Proof of Theorem 2: Suppose that \((R,C = R^T)\) is a non-degenerate symmetric game, where \(R > 0\) without loss of generality. To show that the game has an odd number of symmetric Nash equilibria, we will use the structure of the proof of correctness of the Lemke-Howson algorithm on this game. We first show that the non-degeneracy of the game implies that every vertex of the Lemke-Howson polytope for this game has exactly \(n\) tight constraints, and vice versa.\(^1\)

Lemma 4. \((R,C)\) is non-degenerate if and only if every vertex of the Lemke-Howson polytope for this game has exactly \(n\) tight constraints.

Proof: We start with the forward direction. Let \(z \neq 0\) be a vertex of the Lemke-Howson polytope. There must be at least \(n\) tight constraints at \(z\). Let us define the mixed strategy \(x = \frac{z}{|z|}\). Suppose without loss of generality that \(x_1, \ldots, x_j > 0\) and \(x_{j+1} = x_{j+2} = \cdots = x_n = 0\), where \(j\) could equal \(n\). We will show that exactly \(j\) of the constraints of the form \(R_i z \leq 1\) are tight. To see this, we note that, because of the non-degeneracy of the game, there are at most \(j\) pure best responses to \(x\), namely \(i_1, i_2, \ldots, i_{\ell}\) where \(\ell \leq j\). Because these are the best responses to \(x\) it must be that

\[
R_{i_1} x = R_{i_2} x = \cdots = R_{i_\ell} x > R_r x, \forall r \neq i_1, i_2, \ldots, i_\ell.
\]

Hence,

\[
R_{i_1} z = R_{i_2} z = \cdots = R_{i_\ell} z > R_r z, \forall r \neq i_1, i_2, \ldots, i_\ell.
\]

So there are at most \(j\) tight constraints of the form \(R_i z \leq 1\) at \(z\), and we also know that \(z\) has exactly \(n - j\) tight constraints of the form \(z_i \geq 0\). So there are at most \(n\) tight constraints at \(z\), and because \(z\) is a vertex of the polytope there are exactly \(n\) tight constraints.

For the other direction, we show that, if every vertex of the Lemke-Howson polytope for \((R,C)\) is the intersection of exactly \(n\) hyperplanes, then the game must be non-degenerate. Let \(x\) be a mixed strategy for the column player. Notice that \(x\) can be written as \(\frac{z}{|z|}\) where \(z\) is a point in the polytope. If \(x\) is mixing over \(j\) pure strategies then exactly \(n - j\) of the inequalities \(z_i \geq 0\) are tight at \(z\). But, by our assumption, at most \(n\) constraints can be tight at \(z\), so at most \(j\) inequalities of the form \(R_i z \leq 1\) are tight at \(z\). For any pure strategy \(e_i\) we have that \(e_i^T R x = R_i \frac{1}{|z|} \leq \frac{1}{|z|}\). Since at most \(j\) of these inequalities are tight we get that at most \(j\) pure strategies \(e_i\) best respond to \(x\). So \((R,C)\) is non-degenerate. \(\Box\)

Lemma 4 establishes that we can use the Lemke-Howson algorithm to find a symmetric Nash equilibrium of \((R,C)\). To conclude the proof of Theorem 2, we show that the symmetric Nash equilibria of the game are in one-to-one correspondence with the vertices of the Lemke-Howson polytope that are non-trivial democracies. Since we have already argued that there is an odd number of non-trivial democracies, it follows then that there is an odd number of symmetric Nash equilibria.

Lemma 5. There is a one-to-one correspondence between the symmetric Nash equilibria of \((R,C)\) and the non-trivial democracies of the Lemke-Howson polytope for \((R,C)\).

Proof: We first show that, if \((z,x)\) is a symmetric Nash equilibrium of \((R,C)\), then \(z = \frac{x}{x^T R x}\) is a non-trivial democracy. (Recall that we have assumed that \(R > 0\).) Suppose w.l.o.g. that \(x_1, \ldots, x_l > 0\) and \(x_{l+1} = \cdots = x_n = 0\), where \(l\) may equal \(n\). By the definition of a Nash equilibrium:

\[
\forall i \in \{1, \ldots, l\} : R_i x = x^T R x.
\]

\[
\forall i : R_i x \leq x^T R x
\]

Hence, for all \(i\) s.t. \(z_i > 0\): \(R_i z = 1\); and for all \(i\): \(R_i z \leq 1\). Hence \(z\) is a non-trivial democracy.

To establish a one-to-one correspondence, we prove that the above mapping is onto, and that no two symmetric Nash equilibria map to the same democracy. In the previous lecture, we proved that, if a vertex \(z \neq 0\) of the polytope is a democracy, then \((\frac{z_1}{|z|}, \frac{z_2}{|z|})\) is a symmetric Nash equilibrium of \((R,C)\).

It is easy to see that our mapping maps \((\frac{z_1}{|z|}, \frac{z_2}{|z|})\) to \(z\). So the mapping is onto.

\(^1\)We actually do not need the backward direction for the proof of Theorem 2, but we show it for completeness.
Let us now show that no two symmetric Nash equilibria map to the same democracy. Suppose \((x_1, x_1) \neq (x_2, x_2)\) are symmetric Nash equilibria of \((R, C)\) mapping to the same democracy. It must be that \(\frac{x_1^T R x_1}{x_1^T R x_1} = \frac{x_2^T R x_2}{x_2^T R x_2}\). Since \(x_1^T R x_1, x_2^T R x_2\) are scalars and \(x_1, x_2\) are distributions, it must be that \(x_1 = x_2\), a contradiction. □ □ □

We conclude with two remarks.

**Remark 1.** As a corollary of Theorem 1, we obtain that a non-degenerate 2-player zero-sum game has a unique Nash equilibrium. Indeed, Theorem 1 implies that there is an odd number of Nash equilibria. But we have established that, in two-player zero-sum games, Nash equilibria comprise a convex set. Hence, there must be exactly one equilibrium. Of course, there is a direct way to argue this by looking at the consequences of non-degeneracy on the linear program for computing min-max strategies.

**Remark 2.** Our proof of Theorem 1 went through symmetric games. It is possible to prove the theorem directly by using the general version of the Lemke-Howson algorithm that applies to non-symmetric games. We chose to do the proof via symmetric games for simplicity of exposition, and to avoid a presentation of the general Lemke-Howson algorithm.

**References**