Modulation and Filtering

- Wireless communication application
- Impulse function definition and properties
- Fourier Transform of Impulse, Sine, Cosine
- Picture analysis using Fourier Transforms

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Motivation for Modulation

- Modulation is used to change the frequency band of a signal
  - Enables RF communication in different frequency bands
    - Used in cell phones, AM/FM radio, WLAN, cable TV, ...
  - Note: higher frequencies lead to smaller antennas
Motivation for Filtering

- Filtering is used to remove undesired signals outside of the frequency band of interest
  - Enables selection of a specific radio, TV, WLAN, cell phone, cable TV channel ...
  - Undesired channels are often called interferers
The Fourier Transform as a Tool

- Communication signals are often *non-periodic*
- Fourier Transforms allow us to do modulation and filtering analysis using *pictures*

\[ x(t) \Leftrightarrow X(j2\pi f) \]

Where:

\[ x(t) = \int_{-\infty}^{\infty} X(j2\pi f) e^{j2\pi ft} \, df \]

\[ X(j2\pi f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} \, dt \]
Definition of the Impulse Function

• An impulse of area $A$ at time $t_o$ is denoted as:
  \[ A\delta(t - t_o) \]

• Impulses are defined in terms of their properties
  - Area:
    \[ \int_{-\infty}^{\infty} A\delta(t - t_o) = A \]
  - Fourier Transform:
    \[ A\delta(t - t_o) \Leftrightarrow Ae^{-j2\pi ft_o} \]

- Sampling and convolution properties
  - Shown on the next two slides
Sampling Property of Impulses

- **Multiplication** of an impulse and a continuous function leads to **scaling** of the original impulse.
  - The scale factor corresponds to the sample value of the continuous function at the impulse location.

\[
A\delta(t - t_o)y(t) = Ay(t_o)\delta(t - t_o)
\]
Convolution Property of Impulses

- **Convolution** of an impulse and a function leads to *shifting* and *scaling* of the original function
  - The shift value corresponds to the location of the impulse
  - The scale factor corresponds to the area of the impulse

- Convolution is not limited to impulses
  - 6.003 will explore this in great detail

\[ A\delta(t - t_o) \ast y(t) = Ay(t - t_o) \]
Duality of Multiplication And Convolution

- Multiplication in time leads to convolution in frequency:

\[ x(t)y(t) \Leftrightarrow X(j2\pi f) * Y(j2\pi f) \]

  - This is a key property to understand modulation

- Convolution in time leads to multiplication in frequency:

\[ x(t) * y(t) \Leftrightarrow X(j2\pi f)Y(j2\pi f) \]

  - This is a key property to understand filtering

  • We will defer to 6.003 to give you more details here

- We will use this fact in a few weeks to intuitively show the connection between the Fourier Series and Fourier Transform
Fourier Transform of Cosine Wave

- Two real impulses in frequency needed for cosine in time

\[
x(t) = \int_{-\infty}^{\infty} X(j2\pi f) e^{j2\pi ft} df
\]

\[
= \int_{-\infty}^{\infty} \frac{K}{2} \left( \delta(f+f_o) + \delta(f-f_o) \right) e^{j2\pi ft} df
\]

\[
= \frac{K}{2} \left( e^{-j2\pi f_o t} + e^{j2\pi f_o t} \right)
\]

\[
= K \cos(2\pi f_o t)
\]

\[
\frac{K}{2} \left( \delta(f+f_o) + \delta(f-f_o) \right) \leftrightarrow \frac{K}{2} \cos(2\pi f_o t)
\]

\[
 X(j2\pi f) \rightarrow K/2 \quad f_o = -1/T \quad 0 \quad f_o = 1/T
\]
Fourier Transform of Sine Wave

- Two imaginary impulses in frequency needed for sine in time

\[ x(t) = \int_{-\infty}^{\infty} \frac{jK}{2} \left( \delta(f+f_0) - \delta(f-f_0) \right) e^{j2\pi ft} df \]

\[ = \frac{jK}{2} \left( e^{-j2\pi f_0 t} - e^{j2\pi f_0 t} \right) \]

\[ = \frac{K}{j2} \left( -e^{-j2\pi f_0 t} + e^{j2\pi f_0 t} \right) \]

\[ = K \sin(2\pi f_0 t) \]
• **AM** stands for *amplitude* modulation
  - Frequency and phase modulation are also commonly used

• Key operation is to *multiply* (i.e. *mix*) an input signal with a cosine (or sine) wave
  - This leads to an oscillating waveform whose amplitude varies according to the input signal

• **Analysis:**
  \[ x(t)y(t) \Leftrightarrow X(j2\pi f) \ast Y(j2\pi f) \]
Fourier Transform Allows *Picture Analysis*

\[ y(t) = 2\cos(2\pi f_0 t) \]

\[ x(t) \rightarrow \hat{X}(j2\pi f) \]

\[ z(t) \rightarrow \hat{Z}(j2\pi f) \]

- Input signal is shifted to higher frequency band

\[ X(j2\pi f) \]

\[ f \]

\[ -A \]

\[ 0 \]

\[ -f_0 \]

\[ f_0 \]

\[ Y(j2\pi f) \]

\[ \uparrow 1 \]

\[ -f_0 \]

\[ 0 \]

\[ f_0 \]

\[ Z(j2\pi f) \]

\[ \uparrow 1 \]

\[ -A \]

\[ A^- \]

\[ -f_0 \]

\[ 0 \]

\[ f_0 \]
AM Demodulation (Receiver)

- Input signal is shifted to lower and higher frequency bands
  - Want baseband portion
Apply Lowpass Filter

\[ y(t) = 2\cos(2\pi f_0 t) \]

- High frequency band portion is eliminated with lowpass filter
  - Only baseband portion remains

\[ z(t) \quad w(t) \quad H(j2\pi f) \quad w(t) \rightarrow r(t) \]

\[ W(j2\pi f) \quad H(j2\pi f) \]

\[ -2f_0 \quad -f_0 \quad 0 \quad f_0 \quad 2f_0 \]

\[ -2f_0 \quad -f_0 \quad 0 \quad f_0 \quad 2f_0 \]

Fourier Series and Fourier Transform, Slide 14
Impact of Frequency Offset

- Baseband signal is corrupted!
  - Filtering cannot fix this

\[ y(t) = 2\cos(2\pi(f_o + \varepsilon)t) \]

\[ z(t) \rightarrow w(t) \rightarrow H(j2\pi f) \rightarrow r(t) \]

\[ Y(j2\pi f) = Z(j2\pi f) * 1 \]

\[ W(j2\pi f) = 2A \]

\[ -2f_o - \varepsilon \quad -f_o \quad 0 \quad f_o \quad 2f_o + \varepsilon \]

6.082 Spring 2007
Summary

• The impulse function is an important concept for Fourier Transform analysis
  - Fourier Transforms of cosines and sines consist of impulses
  - Defined in terms of its properties
    • Area, Multiplication (sampling), Convolution

• The Fourier Transform allows picture analysis of modulation and filtering
  - Modulation shifts in frequency (convolution with impulses)
  - Filtering multiplies in frequency

• More details on filtering in next lecture
  - Design of filters in Matlab (for Lab exercises)
    • This tool only works with discrete-time signals
  - Discrete-Time Fourier Transform introduced