Sampling Continuous-Time Signals

- Impulse train and its Fourier Transform
- Impulse samples versus discrete-time sequences
- Aliasing and the Sampling Theorem
- Anti-alias filtering
- Comparison of FT, DTFT, Fourier Series

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The Need for Sampling

- The boundary between analog and digital
  - Real world is filled with continuous-time signals
  - Computers (i.e. Matlab) operate on sequences
- Crossing the analog-to-digital boundary requires sampling of the continuous-time signals
- Key questions
  - How do we analyze the sampling process?
  - What can go wrong?
An Analytical Model for Sampling

- **Two step process**
  - Sample continuous-time signal every $T$ seconds
    - Model as *multiplication* of signal with *impulse train*
  - Create sequence from amplitude of scaled impulses
    - Model as *rescaling* of time axis ($T \rightarrow 1$)
    - Notation: replace impulses with stem symbols

**Can we model this in the frequency domain?**
Fourier Transform of Impulse Train

- Impulse train in time corresponds to impulse train in frequency
  - Spacing in time of $T$ seconds corresponds to spacing in frequency of $1/T$ Hz
  - Scale factor of $1/T$ for impulses in frequency domain
  - Note: this is painful to derive, so we won't ...

- The above transform pair allows us to see the following with pictures
  - Sampling operation in frequency domain
  - Intuitive comparison of FT, DTFT, and Fourier Series
Frequency Domain View of Sampling

- Recall that multiplication in time corresponds to convolution in frequency

\[ x(t)y(t) \iff X(j2\pi f) \ast Y(j2\pi f) \]

- We see that sampling in time leads to a periodic Fourier Transform with period \( \frac{1}{T} \)
Frequency Domain View of Output Sequence

- **Scaling in time leads to scaling in frequency**
  - Compression/expansion in time leads to expansion/compression in frequency

- **Conversion to sequence amounts to** $T \rightarrow 1$
  - Resulting Fourier Transform is now periodic with period 1
  - Note that we are now essentially dealing with the DTFT
Summary of Sampling Process

- Sampling leads to periodicity in frequency domain

We need to avoid overlap of replicated signals in frequency domain (i.e., aliasing)
The Sampling Theorem

- Overlap in frequency domain (i.e., aliasing) is avoided if:
  \[ \frac{1}{T} - f_{bw} \geq f_{bw} \Rightarrow \frac{1}{T} \geq 2f_{bw} \]

- We refer to the minimum \( 1/T \) that avoids aliasing as the Nyquist sampling frequency.
Example: Sample a Sine Wave

• Time domain: resulting sequence maintains the same period as the input continuous-time signal
• Frequency domain: no aliasing

Sample rate is well above Nyquist rate
Increase Input Frequency Further ...

Sample rate is at Nyquist rate

- Time domain: resulting sequence still maintains the same period as the input continuous-time signal
- Frequency domain: no aliasing
Increase Input Frequency Further …

Sample rate is at half the Nyquist rate

- Time domain: resulting sequence now appears as a DC signal!
- Frequency domain: aliasing to DC
Increase Input Frequency Further ...

• Time domain: resulting sequence is now a sine wave with a different period than the input
• Frequency domain: aliasing to lower frequency

Sample rate is well below the Nyquist rate
The Issue of High Frequency Noise

- We typically set the sample rate to be large enough to accommodate full bandwidth of signal.
- Real systems often introduce noise or other interfering signals at higher frequencies.
  - Sampling causes this noise to alias into the desired signal band.
Anti-Alias Filtering

Practical A-to-D converters include a continuous-time filter before the sampling operation:

- Designed to filter out all noise and interfering signals above $1/(2T)$ in frequency
- Prevents aliasing
Using the Impulse Train to Compare the FT, DTFT, and Fourier Series
Relationship Between FT and DTFT

### FT
- **Time:** Continuous, Non-Periodic
- **Freq:** Non-Periodic, Continuous

### DTFT
- **Time:** Discrete, Non-Periodic
- **Freq:** Periodic, Continuous

<table>
<thead>
<tr>
<th>FT</th>
<th>DTFT</th>
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<tbody>
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<td>Time: Continuous, Non-Periodic</td>
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Summary

- The impulse train and its Fourier Transform form a very powerful analysis tool using *pictures*
  - Sampling, comparison of FT, DTFT, Fourier Series

- **Sampling analysis:**
  - **Time domain:** multiplication by an impulse train followed by re-scaling of time axis (and conversion to stem symbols)
  - **Frequency domain:** convolution by an impulse train followed by re-scaling of frequency axis

- **Prevention of aliasing**
  - Sample faster than Nyquist sample rate of signal bandwidth
  - Use anti-alias filter to cut out high frequency noise

- **Up next:** downsampling, upsampling, reconstruction