Segment Intersection

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Problem

• Segment intersection problem:
  – Given: a set of $n$ distinct segments $s_1 \ldots s_n$, represented by coordinates of endpoints
  – Detection: detect if there is any pair $s_i \neq s_j$ that intersects
  – Reporting: report all pairs of intersecting segments
Segment intersection

- Easy to solve in $O(n^2)$ time
- Is it possible to get a better algorithm for the reporting problem?
- **NO** (in the worst-case)
- However:
  - We will see we can do better for the detection problem
  - Moreover, the number of intersections $P$ is usually small.

Then, we would like an *output sensitive* algorithm, whose running time is low if $P$ is small.
Result

• We will show:
  – $O(n \log n)$ time for detection
  – $O((n + P) \log n)$ time for reporting

• We will use *line sweep approach*
  – Many other applications, e.g., Voronoi diagrams, motion planning
Orthogonal segments

- All segments are either horizontal or vertical
- Assumption: all coordinates are distinct
- Therefore, only vertical-horizontal intersections exist
Orthogonal segments

• Sweep line:
  – A *vertical line* sweeps the plane from left to right
  – It “stops” at all “important” x-coordinates, i.e., when it hits a V-segment or endpoints of an H-segment
  – Invariant: all intersections on the left side of the sweep line have been already reported
Orthogonal segments ctd.

- We maintain sorted y-coordinates of H-segments currently intersected by the sweep line (using a balanced BST $V$).
- When we hit the left point of an H-segment, we add its y-coordinate to $V$.
- When we hit the right point of an H-segment, we delete its y-coordinate from $V$. 
Orthogonal segments ctd.

- Whenever we hit a V-segment having coord. \( y_{\text{top}}, y_{\text{bot}} \), we report all H-segments in \( V \) with y-coordinates in \([y_{\text{top}}, y_{\text{bot}}]\).
Algorithm

• Sort all V-segments and endpoints of H-segments by their x-coordinates – this gives the “trajectory” of the sweep line

• Scan the elements in the sorted list:
  – Left endpoint: add segment to tree $V$
  – Right endpoint: remove segment from $V$
  – V-segment: report intersections with the H-segments stored in $V$
Analysis

• Sorting: $O(n \log n)$
• Add/delete H-segments to/from vertical data structure $V$:
  – $O(\log n)$ per operation
  – $O(n \log n)$ total
• Processing V-segments:
  – $O(\log n)$ per intersection
  – $O(P \log n)$ total
• Overall: $O((P + n) \log n)$ time
• Can be improved to $O(P + n \log n)$
Analyzing intersections

• Given:
  – A BST $V$ containing $y$-coordinates
  – An interval $I=[y_{\text{bot}}, y_{\text{top}}]$  

• Goal: report all $y$’s in $V$ that belong to $I$

• Algorithm:
  – $y=\text{Successor}(y_{\text{bot}})$
  – While $y \leq y_{\text{top}}$
    • Report $y$
    • $y:=\text{Successor}(y)$
  – End

• Time: (number of reported $y$’s)*$O(\log n)$ + $O(\log n)$
The general case

- Assumption: all coordinates of endpoints and intersections distinct
- In particular:
  - No vertical segments
  - No three segments intersect at one point
Sweep line

• Invariant (as before): all intersections on the left of the sweep line have been already reported
• Stops at all “important” x-coordinates, i.e., when it hits endpoints or intersections
• Problem I: Do not know the intersections in advance!
• The list of intersection coordinates is constructed and maintained \textit{dynamically}
  (in a “horizontal” data structure $H$)
Sweep line

- Also need to maintain the information about the segments intersecting the sweep line
- **Problem II: Cannot keep the values of y-coordinates of the segments!**
- Instead, we will maintain their **order**. I.e., at any point, we maintain all segments intersecting the sweep line, sorted by the y-coordinates of the intersections (in a “vertical” data structure $V$)
- Key idea: check only for the intersections of segments that are **neighbors** in $V$
Java applet

- Example used in the lecture:
Algorithm

- Initialize the “vertical” BST $V$ (to “empty”)
- Initialize the “horizontal” priority queue $H$ (to contain the segments’ endpoints sorted by x-coordinates)
- Repeat
  - Take the next “event” $p$ from $H$:
    // Update $V$
    - If $p$ is the left endpoint of a segment, add the segment to $V$
    - If $p$ is the right endpoint of a segment, remove the segment from $V$
    - If $p$ is the intersection point of $s$ and $s'$, swap the order of $s$ and $s'$ in $V$, report $p$

(continued on the next slide)
Algorithm ctd.

// Update H
- For each new pair of neighbors $s$ and $s'$ in $V$:
  - Check if $s$ and $s'$ intersect on the right side of the sweep line
  - If so, add their intersection point to $H$
  - Remove the possible duplicates in $H$
- Until $H$ is empty
Analysis

- Initializing $H$: $O(n \log n)$
- Updating $V$:
  - $O(\log n)$ per operation
  - $O((P+n) \log n)$ total
- Updating $H$:
  - $O(\log n)$ per intersection
  - $O(P \log n)$ total
- Overall: $O((P+n) \log n)$ time
Correctness

- All reported intersections are correct
- Assume there is an intersection not reported. Let \( p=(x,y) \) be the first such unreported intersection (of \( s \) and \( s' \))
- Let \( x' \) be the last event before \( p \). Observe that:
  - At time \( x' \) segments \( s \) and \( s' \) are neighbors on the sweep line
  - Since no intersections were missed till then, \( V \) maintained the right order of intersecting segments
  - Thus, \( s \) and \( s' \) were neighbors in \( V \) at time \( x' \). Thus, their intersection should have been detected