Voronoi Diagrams

(Slides mostly by Allen Miu)
Post Office: What is the area of service?

$p_i$ : site points
$q$ : free point
$e$ : Voronoi edge
$v$ : Voronoi vertex
Definition of Voronoi Diagram

- Let $P$ be a set of $n$ distinct points (sites) in the plane.

- The Voronoi diagram of $P$ is the subdivision of the plane into $n$ cells, one for each site.

- A point $q$ lies in the cell corresponding to a site $p_i \in P$ iff $||q-p_i|| < ||q-p_j||$, for each $p_i \in P, j \neq i$. 

March 1, 2005

Lecture 8: Voronoi Diagrams
Demo

http://www.diku.dk/students/duff/Fortune/
http://wwwpi6.fernuni-hagen.de/GeomLab/VoroGlide/
See also the implementation page from Christopher Gold's site www.Voronoi.com.

Enough already!!

Delaunay triangulations and farthest point Delaunay triangulations using 3d convex hulls by Daniel Mark Abrahams-Gessel, fortunately stolen by Anirudh Modi before the original page was taken off the Web. This is the best one!

Convex hulls, Delaunay triangulations, Voronoi diagrams, and proximity graphs by James E. Baker, Isabel F. Cruz, Luis D. Lejter, Giuseppe Liotta, and Roberto Tamassia. Source code is available.

Incremental Delaunay triangulations and Voronoi diagrams by Frank Bossen

Voronoi Diagram/Delaunay Triangulation by Paul Chew uses a randomized incremental algorithm with "brute force" point location.

2-Site Voronoi diagrams by Matt Dickerson, from the Middlebury College Undergraduate Research Project in Computational Geometry

The convex hull/Voronoi diagram applet from the GeomNet project provides a secure Java wrapper for existing (non-Java) code. The applet calls qhull to build its convex hulls and Steve Fortune's sweep2 to build its Voronoi diagrams. A forms interface to the same programs is also available.

VoroGlide, by Christian Icking, Rolf Klein, Peter Köllner, and Lihong Ma. Smoothly maintains the convex hull, Voronoi diagram, and Delaunay triangulation as points are moved, illustrates incremental construction of the Delaunay triangulation, and includes a recorded demo. Now on a faster server!

Delaunay triangulations by Geoff Leach compares several (very) naïve algorithms. Source code is available.

Bisectors and Voronoi diagrams under convex (polygonal) distance functions by Lihong Ma. The diagram is updated on the fly while sites or vertices of the unit ball are inserted, deleted, or dragged around. Very cool!

Delaunay triangulations and Dirichlet tesselations and a short applet-enhanced tutorial by Eric C. Olson

The Voronoi Game by Dennis Shasha. Try to place points to maximize the area of your Voronoi regions.

Higher-order Voronoi diagrams by Barry Schaudt

Tessy, yet another interactive Voronoi/Delaunay demo from Keith Voegele. Java not required.

ModeMap, by David Watson, draws Voronoi diagrams, Delaunay triangulations, natural neighbor circles (circumcircles of Delaunay triangles), and (for the very patient) radial density contours on the sphere. Don't give it more than 80 points.

Delaunay Triangulation from Zhiyuan Zhao's JAVA Gallery of Geometric Algorithms

Delaunay Triangulation Demo at ESSI, Université de Nice/Sophia-Antipolis, France. X terminal required instead of Java. Extremely slow, at least on this side of the Atlantic.
Outline

• Definitions and Examples
• Properties of Voronoi diagrams
• Complexity of Voronoi diagrams
• Constructing Voronoi diagrams
  – Intuitions
  – Data Structures
  – Algorithm
• Running Time Analysis
• Demo
• Duality and degenerate cases
Voronoi Diagram Example:
1 site
Two sites form a perpendicular bisector

Voronoi Diagram is a line that extends infinitely in both directions, and the two half planes on either side.
Collinear sites form a series of parallel lines
Non-collinear sites form Voronoi half lines that meet at a vertex

A vertex has degree $\geq 3$

A Voronoi vertex is the center of an empty circle touching 3 or more sites.
Voronoi Cells and Segments
Voronoi Cells and Segments

Segment

Bounded Cell

Unbounded Cell

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Pop quiz

Which of the following is true for 2-D Voronoi diagrams?

Four or more non-collinear sites are…
1. sufficient to create a bounded cell
2. necessary to create a bounded cell
3. 1 and 2
4. none of above
Which of the following is true for 2-D Voronoi diagrams?

Four or more non-collinear sites are…
1. sufficient to create a bounded cell
2. necessary to create a bounded cell
3. 1 and 2
4. none of above
Degenerate Case: no bounded cells!
Summary of Voronoi Properties

A point $q$ lies on a Voronoi edge between sites $p_i$ and $p_j$ \textbf{iff} the largest empty circle centered at $q$ touches only $p_i$ and $p_j$

- A Voronoi edge is a subset of the locus of points equidistant from $p_i$ and $p_j$

$p_i :$ site points
$e :$ Voronoi edge
$v :$ Voronoi vertex
Summary of Voronoi Properties

A point \( q \) is a vertex *iff* the largest empty circle centered at \( q \) touches at least 3 sites

- A Voronoi vertex is an intersection of 3 more segments, each equidistant from a pair of sites

\( p_i \) : site points
\( e \) : Voronoi edge
\( v \) : Voronoi vertex
Voronoi diagrams have linear complexity \( \{v, e = O(n)\} \)

Intuition: Not all bisectors are Voronoi edges!

- \( p_i \): site points
- \( e \): Voronoi edge
Voronoi diagrams have linear complexity \( \{v, e = O(n)\} \)

Claim: For \( n \geq 3 \), \( v \leq 2n - 5 \) and \( e \leq 3n - 6 \)

Proof: (General Case)

- Euler’s Formula: for connected, planar graphs, 
  \[ v - e + f = 2 \]

Where:

- \( v \) is the number of vertices
- \( e \) is the number of edges
- \( f \) is the number of faces
Voronoi diagrams have linear complexity \( \{v, e = O(n)\} \)

Claim: For \( n \geq 3 \), \( v \leq 2n - 5 \) and \( e \leq 3n - 6 \)

Proof: (General Case)

- For Voronoi graphs, \( f = n \rightarrow (v + 1) - e + n = 2 \)

To apply Euler’s Formula, we “planarize” the Voronoi diagram by connecting half lines to an extra vertex.
Voronoi diagrams have linear complexity \( \{v, e = O(n)\} \)

Moreover,

\[
\sum_{v \in \text{Vor}(P)} \deg(v) = 2 \times e
\]

and

\[
\forall v \in \text{Vor}(P), \quad \deg(v) \geq 3
\]

so

\[2 \times e \geq 3(v + 1)\]

together with

\[(v + 1) - e + n = 2\]

we get, for \( n \geq 3 \)

\[v \leq 2n - 5, \quad e \leq 3n - 6\]
A really degenerate case

- The graph has “loops”, i.e., edges from $p_\infty$ to itself
- The “standard” Euler formula does not apply
- But:
  - One can extend Euler formula to loops (each loop creates a new face) and show that it still works
  - Or, one can recall that the Voronoi diagram for this case has still a linear complexity

\[ \cdots \]
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Constructing Voronoi Diagrams

Given a half plane intersection algorithm…
Constructing Voronoi Diagrams

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Constructing Voronoi Diagrams

Given a half plane intersection algorithm…
Constructing Voronoi Diagrams

Given a half plane intersection algorithm…

Repeat for each site

Running Time:
$O(n^2 \log n)$
Faster Algorithm

• Fortune’s Algorithm:
  – Sweep line approach
  – Voronoi diagram constructed as horizontal line sweeps the set of sites from top to bottom
  – Incremental construction:
    • maintains portion of diagram which cannot change due to sites below sweep line,
    • keeps track of incremental changes for each site (and Voronoi vertex) it “sweeps”
Algorithms Outline

- Ideas
- Data structures
- Events
Invariant

What is the invariant we are looking for?

Maintain a representation of the locus of points \( q \) that are closer to some site \( p_i \) above the sweep line than to the line itself (and thus to any site below the line).
Beach line

Which points are closer to a site above the sweep line than to the sweep line itself?

The set of parabolic arcs form a beach-line that bounds the locus of all such points.
Break points trace out Voronoi edges.
Aches flatten out as sweep line moves down.
Eventually, the middle arc disappears.
Circle Event

We have detected a circle that is empty (contains no sites) and touches 3 or more sites.
Beach Line Properties

• Voronoi edges are traced by the break points as the sweep line moves down.
  – Emergence of a new break point(s) (from formation of a new arc or a fusion of two existing break points) identifies a new edge

• Voronoi vertices are identified when two break points meet (fuse).
  – Decimation of an old arc identifies new vertex
Algorithms Outline

• Ideas
• Data structures
• Events
Data Structures

- Current state of the Voronoi diagram
  - Doubly linked list of half-edge, vertex, cell records
- Current state of the beach line
  - Keep track of break points
  - Keep track of arcs currently on beach line
- Current state of the sweep line
  - Priority event queue sorted on decreasing y-coordinate
Doubly Linked List ($D$)

- Goal: a simple data structure that allows an algorithm to traverse a Voronoi diagram’s segments, cells and vertices
Doubly Linked List ($D$)

- Divide segments into uni-directional half-edges
- A chain of counter-clockwise half-edges forms a cell
- Define a half-edge’s “twin” to be its opposite half-edge of the same segment

Cell($p_i$)
Doubly Linked List ($D$)

- **Cell Table**
  - $\text{Cell}(p_i)$: pointer to any incident half-edge

- **Vertex Table**
  - $v_i$: list of pointers to all incident half-edges

- **Doubly Linked-List of half-edges; each has:**
  - Pointer to Cell Table entry
  - Pointers to start/end vertices of half-edge
  - Pointers to previous/next half-edges in the Cell chain
  - Pointer to twin half-edge
Balanced Binary Tree ($T$)

- Internal nodes represent break points between two arcs
  - Also contains a pointer to the $D$ record of the edge being traced
- Leaf nodes represent arcs, each arc is in turn represented by the site that generated it
  - Also contains a pointer to a potential circle event
Event Queue (Q)

• An event is an interesting point encountered by the sweep line as it sweeps from top to bottom
  – Sweep line makes discrete stops, rather than a continuous sweep

• Consists of Site Events (when the sweep line encounters a new site point) and Circle Events (when the sweep line encounters the bottom of an empty circle touching 3 or more sites).

• Events are prioritized based on y-coordinate
Summarizing Data Structures

• Current state of the Voronoi diagram
  – Doubly linked list of half-edge, vertex, cell records

• Current state of the beach line
  – Keep track of break points
    • Inner nodes of binary search tree; represented by a tuple
  – Keep track of arcs currently on beach line
    • Leaf nodes of binary search tree; represented by a site that generated the arc

• Current state of the sweep line
  – Priority event queue sorted on decreasing y-coordinate
Algorithms Outline

- Ideas
- Data structures
- Events
Circle Event

An arc disappears whenever an empty circle touches three or more sites and is tangent to the sweep line.

Sweep line helps determine that the circle is indeed empty.
Site Event

A new arc appears when a new site appears.
Site Event

A new arc appears when a new site appears.
Site Event

Original arc above the new site is broken into two

$\rightarrow$ Number of arcs on beach line is $O(n)$
Event Queue Summary

• Site Events are
  – given as input
  – represented by the (x,y)-coordinate of the site point

• Circle Events are
  – represented by the (x,y)-coordinate of the lowest point of an empty circle touching three or more sites
  – computed on the fly (intersection of the two bisectors in between the three sites)
  – “anticipated”: these newly generated events may represented by the (x,y)-coordinate of the lowest point of an empty circle touching three or more sites; they can be false and need to be removed later

• Event Queue prioritizes events based on their y-coordinates
“Algorithm”

1. Initialize
   - Event queue $Q \leftarrow$ all site events
   - Binary search tree $T \leftarrow \emptyset$
   - Doubly linked list $D \leftarrow \emptyset$

2. While $Q$ not $\emptyset$, 
   - Remove event $(e)$ from $Q$ with largest $y$-coordinate
     - $HandleEvent(e)$
Handling Site Events

1. Locate the existing arc (if any) that is above the new site
2. Break the arc by replacing the leaf node with a sub tree representing the new arc and its break points
3. Add two half-edge records in the doubly linked list
4. Check for potential circle event(s), add them to event queue if they exist
Handling Circle Events

1. Add vertex to corresponding edge record in doubly linked list
2. Delete from T the leaf node of the disappearing arc and its associated circle events in the event queue
3. Create new edge record in doubly linked list
4. Check the new triplets formed by the former neighboring arcs for potential circle events
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Handling Site Events

1. Locate the leaf representing the existing arc that is above the new site
   - Delete the potential circle event in the event queue
2. Break the arc by replacing the leaf node with a sub tree representing the new arc and break points
3. Add a new edge record in the link list
4. Check for potential circle event(s), add them to queue if they exist
   - Store in the corresponding leaf of T a pointer to the new circle event in the queue

Running Time

- O(log n)
- O(1)
- O(1)
- O(1)
Handling Circle Events

1. Delete from T the leaf node of the disappearing arc and its associated circle events in the event queue
2. Add vertex record in doubly link list
3. Create new edge record in doubly link list
4. Check the new triplets formed by the former neighboring arcs for potential circle events

Running Time

\(O(\log n)\)
\(O(1)\)
\(O(1)\)
\(O(1)\)
Total Running Time

- Each new site can generate at most two new arcs → beach line can have at most $2n - 1$ arcs
- Each “false circle event” can be charged to a real event → $O(n)$ events
- Site/Circle Event Handler $O(\log n)$

$\Rightarrow O(n \log n)$ total running time
Is Fortune’s Algorithm Optimal?

- We can sort numbers using any algorithm that constructs a Voronoi diagram!

- Map input numbers to a position on the number line. The resulting Voronoi diagram is doubly linked list that forms a chain of unbounded cells in the left-to-right (sorted) order.
Remaining slides for the curious
A Circle Event

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Lecture 8: Voronoi Diagrams
Add vertex to corresponding edge record

Half Edge Record
Endpoints.add(x, y)

Link!
Deleting disappearing arc

\[
< p_j, p_k >
\]

\[
< p_i, p_j >
\]

\[
< p_m, p_l >
\]

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Deleting disappearing arc

\[ \langle p_j, p_k \rangle \]

\[ \langle p_i, p_j \rangle \]

\[ \langle p_k, p_m \rangle \]

\[ \langle p_m, p_l \rangle \]
Create new edge record

A new edge is traced out by the new break point \(<p_k, p_m>\)

New Half Edge Record
Endpoints.add(x, y)
Check the new triplets for potential circle events

$p_i < p_j, p_k >$

$p_i < p_j, p_k >$

$p_k < p_m, p_l >$

$p_m < p_l, Q >$

$p_i$, $p_j$, $p_k$, $p_m$, $p_l$ new circle event
Minor Detail

- Algorithm terminates when $Q = \emptyset$, but the beach line and its break points continue to trace the Voronoi edges
  - Terminate these "half-infinite" edges via a bounding box
Algorithm Termination

\[ p_i, p_j, p_k, p_m, p_l \]

\[ Q = \emptyset \]
Algorithm Termination

\[ p_i < p_j, p_m \]
\[ p_m < p_i, p_j \]
\[ p_k < p_m, p_l \]

\[ Q \]

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Algorithm Termination

Terminate half-lines with a bounding box!
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Degenerate Cases

- Events in Q share the same y-coordinate
  - Can additionally sort them using x-coordinate
- Circle event involving more than 3 sites
  - Current algorithm produces multiple degree 3 Voronoi vertices joined by zero-length edges
  - Can be fixed in post processing
Degenerate Cases

• Site points are collinear (break points neither converge or diverge)
  – Bounding box takes care of this

• One of the sites coincides with the lowest point of the circle event
  – Do nothing
Site coincides with circle event: the same algorithm applies!

1. New site detected
2. Break one of the arcs an infinitesimal distance away from the arc’s end point
Site coincides with circle event
Locate the existing arc that is above the new site

- The x coordinate of the new site is used for the binary search
- The x coordinate of each breakpoint along the root to leaf path is computed on the fly
Break the Arc

Corresponding leaf replaced by a new sub-tree

Different arcs can be induced by the same site!
Add a new edge record in the doubly linked list

Endpoints $\leftarrow \emptyset$

Pointers to two half-edge records

New Half Edge Record

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Checking for Potential Circle Events

- Scan for triple of consecutive arcs and determine if breakpoints converge
  - Triples with new arc in the middle do not have breakpoints that converge
Checking for Potential Circle Events

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  - Triples with new arc in the middle do not have breakpoints that converge
Checking for Potential Circle Events

- Scan for triple of consecutive arcs and determine if breakpoints converge
  - Triples with new arc in the middle do not have breakpoints that converge
Converging break points may not always yield a circle event

- Appearance of a new site before the circle event makes the potential circle non-empty

(The original circle event becomes a *false alarm*)
Handling Site Events

1. Locate the leaf representing the existing arc that is above the new site
   - Delete the potential circle event in the event queue
2. Break the arc by replacing the leaf node with a sub tree representing the new arc and break points
3. Add a new edge record in the doubly linked list
4. Check for potential circle event(s), add them to queue if they exist
   - Store in the corresponding leaf of T a pointer to the new circle event in the queue