The Dynamics of Running

Russ Tedrake

February 27, 2007

Announcements:

• PS2 out today
• Looking for stunt pilot

Goals for this lecture

• Walking vs. Running
• Some observations on running animals (comparative biomechanics)
• Raibert Hoppers
• SLIP model
• Koditschek’s simplified hopper
• LLS model

1 Introduction

Definition of running. Simple definition uses the existence of an aerial phase. Better definition (animals can run w/o an aerial phase) looks at the trajectory of the center of mass. In walking, vault over a stiff leg (or legs) acting like an inverted pendulum; the COM is reaches it’s highest point in mid-stance having exchanged kinetic energy into gravitational potential. In running, the COM falls to it’s lowest position at mid-stance. Another way to say this is that in running, the kinetic energy is in phase with the gravitational potential. In walking, it is out of phase. Groucho-running.

2 Comparative Biomechanics

Muybridge helped reconcile a debt by shooting images to prove existence of an aerial phase. I recommend taking a look at his books[6]. Some say that this analysis initiated the study of gait. Studying a single animal in an attempt to find the underlying strategies
is difficult. Some animals are difficult to measure, ... Comparative biomechanics looks for common behaviors across species.

Humans run with one stance leg. Horses have many gaits, but often have more than one stance leg. Insects (such as cockroaches) have tripod gaits. Crabs run sideways. The kinematics and energetics of each of these is very different. But, if you look at the travel of the center of mass, relative to the travel of the center of pressure (which defines the “virtual leg”), then the dynamics of these systems in incredibly similar in all of these animals. The dynamics of all of these animals (based on COM tracking and force plate measurements of ground reaction forces) behave as if the animal has a virtual spring on it’s virtual leg (draw cartoon on the board). This represents a huge dynamic range, from 1e-3kg (cockroach) to 135kg (horse) - that’s five orders of magnitude. The field of many other quantities that are similar - we all tend to transition from walking to running at approximately the same ratio of kinetic to potential energy (the dimensionless Froude number), ...

Another interesting note about this virtual spring. We think that it’s actually a spring (using tendons, not muscles). Evidence includes:

- Running efficiency (measured by volume of oxygen consumed) far surpasses what would be expected based on the efficiency of muscle.
- The is evidence of preflexes - stabilizing forces which occur at a time scale faster than neuromuskular dynamics.
- We find big springs in animals like kangaroos.

Evidence and videos here (cockroach on fractal blocks, etc).

3 Raibert hoppers

Somewhat in parallel, or even preceding, much of this biomechanics analysis, Marc Raibert’s Leg Laboratory was developing a hugely successful series of hopping (running) robots. Show pic/vids of planar one-leg hopper. Show simulation.

Basic control ideas. State machine. Underactuated, but intermittent control opportunities. Control hopping height, forward speed, and body attitude independently. Clever mechanical design made this valid enough.

Videos of the rest of the hoppers. Running on four legs as though they were one.

4 Spring-loaded inverted pendulum (SLIP)

The model is a point mass, \( m \), on top of a massless, springy leg with rest length of \( l_0 \), and spring constant \( k \). The state of the system is given by the \( x, y \) position of the center of mass, and the length, \( l \), and angle \( \theta \) of the leg. Like the rimless wheel, the dynamics are modeled piecewise - with one dynamics governing the flight phase, and another governing the stance phase.
4.1 Approximate solution

This analysis is from [2]. Harmut is currently at MIT as a postdoc.

5 Koditschek’s Simplified Hopper

6 Lateral Leg Spring (LLS)

Running by ricocheting.

Show videos of cockroach. Mechanical response is faster than the monosynaptic reflex, which supports the mechanical response.

References


Flight phase:
- Ballistic trajectory of COM.
- Leg rotates to desired touchdown angle \( \alpha_0 \) and rest length \( l_0 \).
- Tzu has no effect on mass trajectory.

Stance phase:
- Starts when massless toe hits the ground (no collision).
- Ends when spring returns to rest length.
- Assumes no slip.

Goal: understand long-term dynamics of the gait.
Notice: Energy is conserved (should we expect only Lyapunov stability?)

Return map: (apex to apex)
Can understand entire dynamics by studying one-dimensional map from \( y_k \) to \( y_{k+1} \) at the apex.

Why?
- At apex \( \dot{y}_k = 0 \).
- \( \dot{y}_k \) can be expressed in terms of \( y_k \) and the constant \( \frac{\text{total}}{\text{energy}} \).
- \( x_k \) has no effect on dynamics.
- \( x_k = \theta_0, \quad \theta = \phi_0 \), by definition.
Derive apex-to-apex map:

\[ y_k \rightarrow y_{k+1} \]

\[ x_k(y_k) = \frac{1}{\sqrt{\frac{1}{2} m x_k^2 + mg y_k - E}} \]

\[ x_k = \sqrt{E - g y_k} \]

\[ x_k = \sqrt{z} \]

\[ \dot{x}_k = \sqrt{2E - 2gz y_k} \]

At touchdown:

\[ y_{td} = l_0 \sin \alpha_0 \]

\[ z_{td} = x_k \]

\[ \dot{y}_{td} = \sqrt{2E - 2gz \sin \alpha_0 - x_k^2} \]

Stated dynamics:

\[ x_{td}, y_{td}, \dot{y}_{td} \] as a function of 

\[ x_{td}, y_{td}. \]

Write eqs of motion:

Switch to polar form:

\[ r = \sqrt{x^2 + y^2} \]

\[ L = \frac{1}{2} m r^2 + \frac{1}{2} m r^2 \dot{\theta}^2 - \frac{1}{2} k (l_0 - r)^2 - mg r \sin \theta \]

Yields eqs which are known to be non-integrable (w/ elementary functions).
Let's make a small-angle approximation. Around vertical, 

$$\sin \theta \approx 1.$$ 

This yields 

$$L = \frac{1}{2} mr^2 + \frac{1}{2} I \dot{\theta}^2 - \frac{1}{2} k (l_0 - r)^2 - mg r.$$ 

$$\frac{dL}{dr} = mr \dot{\theta}^2 + k (l_0 - r) - mg.$$ 

$$\frac{dL}{\dot{\theta}} = 0.$$ 

$$\frac{dL}{\dot{r}} = mr \dot{\theta}$$ 

$$\frac{d}{dt} \frac{dL}{dr} = mr \ddot{\theta} - k (l_0 - r) + mg.$$ 

$$\frac{d}{dt} \frac{dL}{\dot{\theta}} - \frac{dL}{\dot{r}} = 0 = mr \ddot{\theta} + 2mr \dot{\theta} \dot{r} \quad \Rightarrow \quad \frac{d}{dt} (mr^2 \dot{\theta}).$$ 

$$\ddot{\theta} = -\frac{2mr \dot{r}}{mr^2 \dot{\theta}}.$$ 

$$\dot{r} = r \dot{\theta}^2 + \frac{k}{m} (l_0 - r) - g.$$ 

We would like to integrate to find the mapping.

\[
\begin{bmatrix}
\dot{r} \\
\dot{\theta} \\
\dot{\varphi}
\end{bmatrix}_{\text{to}} = \begin{bmatrix}
\dot{r} \\
\dot{\theta} \\
\dot{\varphi}
\end{bmatrix}_{\text{to}}.
\]
Some of these components are easy, thanks to
- radial symmetry. \(( r_0 = \omega \) at \( r = l_0 \)).

\[ r_0 = \omega \]

\[ \dot{\omega} = -\omega \]

- conservation of angular momentum \( \dot{\omega} = 0 \).

\[ \dot{\tau}_r = \dot{\tau}_r \]

**Constitution of energy.**

Difficult part is \( \dot{r}_0 \). Luckily, due to \( \dot{\omega} = 0 \), we can solve
for \( r(\tau) \), then integrate \( \dot{\omega} = \frac{\omega}{mr^2} \).

Energy, only in terms of \( r(\tau) \) (and constants):

\[ E = \frac{m^2 r^2}{2} + \frac{\omega^2}{2mr^2} + \frac{k}{2}(l_0 - r)^2 + mg \]

Details in Beyer 05,
still have to assume that

\[ |r - l_0| < l_0. \]

\( \rightarrow \) apex map.

Result is a map

\[ Y_{ik} = P(Y_{ic}) \]

Energy correction. Small angle assumption violates
energy conservation if \( Y_{ik} \neq Y_{ic} \).

Correct error by adding back into \( \dot{\omega} \) (happens
automatically by computing \( \dot{\omega} \) from \( Y_{ic} \)).
Kaditschek model.

- Vertical hopping only.
- Mass in the body + mass in the toe.
- Spring model of air piston (consider \( k(f_t - f) \) and \( k(\frac{1}{f}) \)).
- Mass in the toe results in two (inelastic) collisions:
  - toe w/ ground.
  - spring hits spring stop.

Passive system slowly loses energy and stands still.

Consider Rabiet controller:

\[
\text{Flight} \rightarrow \text{Compression} \rightarrow \text{Thrust} \rightarrow \text{Decompression} \downarrow \text{push w/ constant force for fixed duration.}
\]

Again, stable apex return map.

(But this time due to dissipation).

- Linear spring.
- Non-linear spring.