We have seen various notions of fairness. The pragmatic model of congestion control protocol suggest that in an idealized setup rate allocation is done as per fair resource allocation Classically, in a network our interest is in performance. The two main performance metrics are throughput and delay. Here, we examine throughput performance of the network.

Model. Given a network of node with network-connectivity graph $G = (V, E)$. Each edge $e \in E$ has capacity $C_e > 0$.

Let $R$ be set of possible routes; pre-determined. Let $M$ be routing matrix.

$$M = [M_{ei}]_{(E) \times (R)}$$

where $M_{ei} = \begin{cases} 1 & \text{if } iR \text{ passes through } e \in E \\ 0 & \text{otherwise} \end{cases}$
The Resource Allocation problem (RAE) is:

$$\text{max } \sum U(x_i)$$

s.t. $$y = Mx \leq C = (C_e)$$

$$x \geq 0.$$

We consider a dynamic model. Let requests/flows of type $i \in \mathbb{R}$ arrive according to a Poisson process of rate $\lambda_i$. Let they bring in amount of work which is distributed like exponential distribution of $\mu_i$. Let $\phi_i = \lambda_i / \mu_i$. Then $\phi_i$ is average amount of "work" coming to flow type $i \in \mathbb{R}$.

Let $N_i(t)$ be number of flows of type $i$ that are in the system at time $t$. Then each flow of type $i$ is allocated rate $x_i(t)$, where $x_i(t)$ is the solution to the following optimization problem:
\[
\max \sum n_i(t) U(x_i(t)) \\
\text{subject to } z_i(t) = n_i(t) \cdot x_i(t) \leq C \\
x(t) \geq 0.
\]

**Goal:** What are the precise conditions on \( g = (g_i) \) that allows for \( n(t) \) to remain finite with probability 1.

**Necessary condition.**

\[
M g \leq C
\]

or

\[
\sum_{i \in e} g_i \leq C e.
\]
Sufficient condition.

The first question that comes to one's mind is that if we allocate bandwidth to flows maximally, can it happen that the above stated necessary conditions are not sufficient?

Consider the following counter-example:

There are 3 links with unit capacity and 4 flow types: 0, 1, 2, 3 pass through them as shown above. Consider priority allocation: if flow \( i \in \{1, 2, 3\} \) is present, it gets all of link \( i \). When all the three links are available flow 0 gets the links 1, 2, 3 to itself.
Here, the necessary condition is:

\[ 8_0 + 9_i < 1 \quad \text{for } i = 1, 2, 3 \]  

The priority policy requires

\[ 8_0 < (1-9_1)(1-9_2)(1-9_3) - c_1 \]

in addition to above. Consider,

\[ 8_0 = 9_1 = 9_2 = 9_3 = \frac{1}{3} \]

It clearly satisfies the condition \( c_0 \).
But it does not satisfy \( c_1 \) as

\[ \frac{1}{3} > (2\frac{1}{3})^3 = \frac{8}{27} \]

Thus, we need to understand how the fair allocation policies perform.
Sufficiency of necessary condition for $\alpha$-fair allocation.

Now, we will establish that for any $\alpha \in (0,1)$, the $\alpha$-fair rate allocation policy induces flow numbers to be finite with probability 1 as long as the necessary conditions are satisfied.

Markov chain of number of flows:

Let $(N_i(t))_{t \in \mathbb{R}}$ be number of flows of different types at time $t$.

Let $(\lambda_i(t))_{t \in \mathbb{R}}$ be rate allocated to flow of type $i \in \mathbb{R}$. They are solved to

$$\max_{i \in \mathbb{R}} \sum_{i \in \mathbb{R}} \lambda_i(t) N_i(t)$$

s.t. $$\sum_{i : i \in \mathbb{R}} \lambda_i(t) N_i(t) \leq C_e$$

$i \in \mathbb{R}$

$N_i(t) \geq 0$. 
The above defines Markov chain \((N_i(t))=n(t)\)
This is because:

* Arrival process of flow type \(i\)
  - is a Poisson of rate \(\lambda_i\)

* Departure of flow happens when it is served completely. But the service requirement is exponentially distributed; hence memory-less. The exponential service has mean \(\mu_i\) or rate \(\frac{1}{\mu_i}\). The rate \(\lambda_i\) remains the same till next transition. Hence, effective departure "rate" is \(\frac{\lambda_i}{\mu_i}\).

Thus, Markov chain transition rates are:

\[
N_i(t) \rightarrow N_i(t)+1 \quad \text{at rate} \quad \lambda_i
\]

\[
n_i(t) \rightarrow (n_i(t)-1)^+ \quad \text{at rate} \quad \frac{n_i(t) \cdot \lambda_i}{\mu_i}
\]
We want to show that

\[
\max_{i \in \mathbb{R}} n_i(t) < \infty \text{ w.p. 1.}
\]

For this, as before, we will study the embedded discrete-time Markov chain in which a transition happens every time.

As noted earlier, as long as the rate of transitions is bounded above in the original Markov chain in continuous time, it is sufficient to prove the finiteness of \( N(t) \) with respect to discrete time. Since both \( N(t) \) of transitions and time go to infinity simultaneously. This is true, because arrival rate is finite; service rate \( \bar{\lambda}_i(t) n_i(t) \leq (\max \lambda_i)(\max \frac{1}{\mu_i}) < \infty \).

Therefore, we can study the discrete-time embedded HMC.

To this, next we consider a discrete time Markov chain.
Discrete-time HMC.

State: \( \mathbf{n}(z) = (n_i(z)) ; n(0) = 0 \):

In each time, exactly one transition happens. The transition probabilities are as follows:

\[
P_{n_i, n_i+1} \propto \lambda_i \]

\[
P_{n_i, n_i-1} \propto \frac{x_i n_i}{\mu_i} \]

We will establish that

\[
\max_i n_i(z) < \infty \text{ for all } z \in \mathbb{N}. \]

For this, it is sufficient to establish that \( n(z) \) is positive recurrent as we know.
Theorem. The HMC $n(t)$ is positive recurrent as long as

$$M_P < C.$$

Proof. We will use Lyapunov function Foster's criteria to prove this theorem.

Consider $\alpha$-fair utility.

$$U(x) = \frac{x^{1-\alpha}}{1-\alpha}.$$

The rate allocation at time $t$:

$$\max_{x \geq 0} \sum_{i} n_i(t) \left( \frac{x_i^{1-\alpha}}{1-\alpha} \right)$$

subject to

$$\sum_{i: i \in e} n_i(t) x_i \leq C e + e \in E.$$
Let $\Delta_i = n_i(t) \cdot x_i \Rightarrow x_i = \frac{\Delta_i}{n_i(t)}$

Then:

$$\max \sum \frac{\Delta_i^{1-\alpha}}{n_i(t)^{1-\alpha} \cdot (1-x)}$$

s.t. $\sum \Delta = 1$.

$$\equiv \max \sum \left( n_i^{\alpha}(t) \cdot \Delta_i^{1-\alpha} \right) \cdot \frac{1}{1-\alpha}$$

Define $G(u) = \sum n_i^{\alpha}(t) \cdot \frac{u_i^{1-\alpha}}{1-\alpha}$.

The rate allocation vector $\Delta(t)$ is $\max G(u)$ over convex constraint.

We know that $\Delta(t)$ is unique due to concavity of $G(u)$. 
Now consider any \( \mathbf{u} \) that is feasible null allocation.

Then: by “gradient” condition:
\[
G'(\Lambda) \top (\mathbf{u} - \Lambda) \leq 0.
\]

By concavity:
\[
G'(\mathbf{u}) \top (\mathbf{u} - \Lambda) \leq G'(\Lambda) \top (\mathbf{u} - \Lambda) \leq 0.
\]

[Imagine single variable function: \( f(\cdot) \) which is concave; i.e., \( f''(\cdot) \leq 0 \); Now
\[
f'(x) - f'(y) \approx f''(y)(x-y)
\]
\[
\left( f'(x) - f'(y) \right) (x-y) \approx f''(y)(x-y)^2 \leq 0.
\]

Under the condition \( M \mathbf{g} < \mathbf{c} \), \( \exists \varepsilon > 0 \) such that
\[
(1+\varepsilon) \mathbf{g} \cdot M < \mathbf{c}. \quad \text{That is,}
\]
\[
\mathbf{u} = (1+\varepsilon) \mathbf{g} \text{ is a feasible vector.}
\]
Using above inequality about \( G \) with 
\[ u = (1+\varepsilon) s, \]
we have

\[
\left(G'((1+\varepsilon) s)\right)^T \cdot (1+\varepsilon)s - \Lambda) \leq 0
\]

Now:

\[
\left(G'_{(u)}\right)_{e_i} = \left(n_i^\alpha(t) \cdot u_i^{-\alpha}\right)
\]

Therefore we have

\[
\sum_{i} n_i^\alpha(t) \cdot \beta_i^{-\alpha} (1+\varepsilon)^{-\alpha} \cdot ((1+\varepsilon) \beta_i - \Lambda_i) \leq 0
\]

\[
\Rightarrow \sum_{i} \beta_i^{-\alpha} n_i^\alpha(t) \cdot ((1+\varepsilon) \beta_i - \Lambda_i) \leq 0.
\]
Now, consider Lyapunov function:

\[ V(n(\tau)) = \sum_{i} \frac{n_i^{\alpha+1}(\tau)}{\alpha+1} \cdot \frac{\chi_i}{\tau^{\alpha}} \cdot \mu_i \]

For ease of writing: let's understand the "differentiation" of \( V(n(\tau)) \) on "average":

\[ \frac{d V(n(\tau))}{d\tau} = \sum_{i} n_i^\tau(\tau) \cdot \frac{\chi_i}{\tau^{\alpha}} \cdot \mu_i \cdot d\eta_i(\tau). \]

But, \( \mathbb{E}[d\eta_i(\tau)] = \varphi \cdot \left[ \lambda_i - \Lambda i / \mu_i \right] ; \ \varphi > 0. \)

Therefore:

\[ \mathbb{E} \left[ \frac{d V(n(\tau))}{d\tau} \right] = \varphi \sum_{i} n_i^\tau(\tau) \cdot \frac{\chi_i}{\tau^{\alpha}} \cdot \mu_i \cdot \left[ \lambda_i - \Lambda i / \mu_i \right] \]
\[
\varphi \sum_i n_i^x (z) \cdot g_i^{-\alpha} \cdot \left[ p_i - \Delta_i \right]
\]

\[
= -\varepsilon - \varphi \left[ \sum_i n_i^x (z) \cdot g_i^{-\alpha} \right] + \sum_i n_i^x (z) \cdot g_i^{-\alpha} \left[ \frac{\partial}{\partial z} \right]_{\Delta_i}
\]

But second term \( \leq 0 \) as per above arguments.

Therefore:

\[
E \left[ \frac{dV(m(z))}{d\tau} \right] < -\varepsilon \quad \text{if} \quad n(z) \geq \beta
\]

for some constant \( \beta > 0 \).

This, along with Foster's criteria implies the desired result of positive recurrence of \( n(z) \). \( \text{QED} \)