Fluid model

Outline:

1. Fluid model:
   (a) Scaling
   (b) Equations
   (c) Implications

2. Fluid model: justification

3. Critical fluid models.
Fluid model scaling: an example.

Consider a discrete-time queue.

Arrival process is time i.i.d. Let $A(t)$ denote number of packets arriving to queue in time $t$. We assume that

$$P(A(t) = 2) = p; \quad P(A(t) = 0) = 1 - p.$$

$$E[A(t)] = 2p.$$

Service process is deterministic with a unit packet served at a time. We will assume

$$2p < 1.$$ 

Dynamics: \quad $Q(t+1) = (Q(t) - 1)^+ + A(t).$

Fluid scaling: consider $n \in \mathbb{N}$. 

Define $Q_n(t) = Q(nt) \cdot \frac{1}{n}$, where $Q(nt)$ is extended for all time by interpolation.
\[ Q(nt) = (nt-Lnt)Q(Lnt) + (nt+1-Lnt)Q(Lnt+1). \]

Interest: study \( q^n(\cdot) \) as \( n \to \infty \) over \([0,T]\) for some finite \( T \).

"Expected" result: for every sub-sequential limit, the limiting system follows deterministic dynamics. Specifically,

\[ \mathbb{P}\left( \sup_{t \in [0,T]} |q^{n_i}(t) - q(t)| > \varepsilon \right) \to 0 \quad \text{for any } \varepsilon > 0 \text{ as } n_i \to \infty. \]  

Here,

\[ q(t) = (q(0) - (1-2p)t)^+ \quad \text{for } t \geq 0. \]

More generally:

\[ q(t) = q(0) + \alpha(t) - d(t) \]

\[ \alpha(t) = 2pt; \quad d(t) = \int_{\{s \in [0,t] : q(s) > 0\}} ds \]
Implications:

1. There exists \( t_0 \) such that for \( t \geq t_0 \), \( q(t) = 0 \), if \( q(0) = 1 \).

2. 
\[
q^n(t) = q^n(0) + a^n(t) - a^n(t) \\
\leq q^n(0) + a^n(t) \\
\leq q^n(0) + 2t \quad \text{w.p.} \ 1.
\]

Let, \( q^n(0) = 1 \) for all \( n \) (w.l.o.g.).

Then \( q^n(t) \leq 1 + 2t \leq 1 + 2t \) for \( t \in [0, t] \) w.p. 1.

Therefore, \( q^n(t) \) is a sequence of r.v. that is uniformly integrable (UI).

That is, \( \varphi_n(\theta) = \mathbb{E} \left[ X_n \mathbb{1}_{\{ |X_n| > \beta \}} \right] \)

\[ \sup_n \varphi_n(\beta) \rightarrow 0 \quad \text{as} \quad \beta \rightarrow \infty. \]
3. Due to u.i. and \( q^{n_i} \Rightarrow q \) in prob. we have that

\[
\mathbb{E} [q^{n_i}(t)] \rightarrow \mathbb{E} [q(t)] = q(t)
\]

Take \( t = t_0 \), then for \( n > n_0 \) we have

\[
\mathbb{E} \left[ q^n(t_0) \right] \leq \frac{1}{2}
\]

\[
\mathbb{E} \left[ Q(n,t_0) \right] \leq \frac{n}{2}
\]

\[
\mathbb{E} \left[ Q(n,t_0) - n \right] \leq -\frac{n}{2}
\]

Let \( V(t) = Q(t) \).

Then,

\[
\mathbb{E} \left[ V(V(0) t_0) - V(0) \right] \leq -\frac{V(0)}{2} + C \mathbb{E} [\xi_x e^B].
\]

This will imply Foster-style stability.
**Generalized Foster's Criteria:**

Let $V : X \to \mathbb{R}_+$ be a Lyapunov function with conditions:

$V(0) = 0$; \( \{ x : V(x) \leq N \} \) be bounded for all $N$; \( \mathbb{E} \left[ V(t+1) - V(t) \right] < \infty \) for all $t$; and for constant $C$,

\[
\mathbb{E} \left[ V(c \cdot z) - V(0) \right| V(0) = z] \leq -\frac{V(0)}{2} + C \cdot \mathbb{1}_{z \notin B}
\]

for some bounded set $B$. Then, set $B$ is a positive recurrent.

Thus, fluid model implies stability or positive recurrence of Markov chain.
Fluid model for network:

Let,

\[ Q(t) = (Q(t-1) - S(t-1))^+ + A(t-1). \]

Scaling:

\[ q^n(t) = \frac{1}{n} Q(nt), \text{ where} \]

\[ Q(-) \text{ is defined for all time with linear interpolation as before.} \]

Model:

\[ q(t) = q(0) + a(t) - s(t) + z(t) \]

\[ a(t) = \lambda t \]

\[ s(t) = \sum_{\pi \in S} s_\pi(t). \pi \]

\[ Z(s) = \int_0^s \sum_{i \in S} s_i(s) 1_{\{q_i(s) = 0\}} ds \]
$S_{p}(t) = 0 \quad \text{if} \quad q(t) \cdot \pi < q(t) \cdot \sigma \quad \text{for some} \quad \sigma \neq \pi \in \mathcal{S}.

(for almost all t).

This last equation captures the effect of max-wt scheduling algorithm.
Justification. Here, a sketch of fluid model justification is described.

Let, \( \mu^n_T \) denote probability distribution of scaled system \( q^n(.) \) over \([0,T]\) for some finite \( T \).

Our interest is in limit points of this probability distribution. Specifically, we want to show that any limit point of sequence \( (\mu^n_T)_{n \geq 1} \) has support which is subset of the sample-paths that satisfy fluid model equations.

For this, two steps need to be performed.

**STEP 1.** Show that limit points of \( \mu^n_T \) are valid probability distribution (i.e. nothing bizarre happens in the limit).
STEP 2. Show that under any limit point, the sample path satisfies fluid model equations with probability 1.

Preliminaries. We need to define metric on space of distribution over trajectories of network to make notion of convergence precise.

All queuing trajectories (say, $q^n_i(\cdot)$) over interval $[0, T]$ are continuous (due to linear interpolation). Let $C^N[0, T]$ be set of N dimensional continuous trajectories over $[0, T]$ taking values in $\mathbb{R}^N_+$. Equivalently,

$$C^N[0, T] = \{ F : F : [0, T] \to \mathbb{R}^N_+ ; F \text{cont} \}$$

Let,

$$d(F, G) = \sup_{0 \leq t \leq T} \left| F(t) - G(t) \right|$$

for $F, G \in C^N[0, T]$. 
Modulus of

\[ W_8(F) = \sup_{|s-t| \leq \delta} \left| F(s) - F(t) \right| \]
\[ s, t \in [0, t] \]

Assumption.

\[ q^n(0) \to q(0) \text{ w.p. 1,} \]

so that \[ ||q(0)|| \leq B \]

continuity

Towards STEP 1.

Fact 1. The sample paths \( q^n(t) \) over \([0, T]\) are Lipschitz continuous. This is because at most one packet arrives or departs from any queue in a unit time-slot.

Therefore, \[ W_8(q^n) \leq C \cdot \delta \text{ for some constant } C. \]
Fact 2.

For \( t \in [0, T] \): \[ \| q^n \| = \sup_{0 \leq t \leq T} | q^n(t) | \leq C_2 \cdot T \]

for some constant \( C_2 \). This is because at most 1 arrival per queue per time-slot.

Fact 3.

For arrival process, it is Lipschitz continuous as well as number of arrivals in \([0, T]\) is bounded above by \( C_3 \cdot T \).

Fact 4.

\( s^n(\cdot), d^n(\cdot), z^n(\cdot) \) for similar reason are uniformly bounded and Lipschitz continuous over \([0, T]\).
Facts 1-3 imply the following:

The system descriptor

\[ x^n(t) = (q^n(t), a^n(t), s^n(t), d^n(t), z^n(t)) \]

is uniformly bounded over \([0, T]\) and Lipschitz continuous.

By Thm 7.3 [Bill], the sequence \(\mu^n(t)\) is "tight".

By Thm 5.1 [Bill], the sequence \(\mu^n(t)\) is relatively compact.

That is, every subsequence of \(\mu^n(t)\) has a convergent subsequence which is a valid probability distribution on \(CN[0, T]\).

Thus, all limit points of \(\mu^n(t)\) are valid probability distributions.
REFERENCE:


Towards Step 2.

Consider any subsequence that is convergent. Say $\mu^{n_k} \to \mu$.

Now $C^N[0,1]$ is a complete and separable metric space under metric defined before (i.e. $d(F,G) = \sup_{0 \leq t \leq T} |F(t) - G(t)|$).

By Skorohod’s representation theorem, (Thm 6.7 [Bill]), there exists a common probability space and over that a sequence of random variables $X^{n_k}$, $X$ so that
$X^n_k \sim \mu^n_k$; $X \sim \mu$, and

$X^n_k \rightarrow X$ with probability 1.

Let, $\Omega = \{ \omega : X^n_k(\omega) \rightarrow X(\omega) \}$; $P(\Omega) = 1$.

Now, detailed but simple analysis will show that for every $\omega$, the difference equation of evolution for $n^k$ system give rise to differential equations as $k \rightarrow \infty$. That is, $X(\omega)$ satisfies some differential equations which have desired fluid model dynamics.

We skip details here.

**Conclusion:**

Most of such discretized well-behaved network systems will have fluid model. However, one needs to be careful in checking "details".