Markov Chains

**Definition.** Let \( X_1, \ldots, X_n, \ldots \) be a sequence of random variables taking values in some finite or countably finite space \( E \) such that

\[ P_{ij} = P(X_{n+1} = j \mid X_n = i) = P(X_{n+1} = j \mid X_n = i, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0) \]

for all \( i, j \in E, n \geq 0 \) and \( (i_0, \ldots, i_n) \). Then \( \{X_n\}_{n \geq 0} \) is called a time homogeneous Markov chain (CHMC).

The matrix \( P = [P_{ij}] \) is called its transition matrix.

We will denote "history" \( \{X_1, \ldots, X_n\} \) by \( F_n \). That is, \( F_n \) contains information about past up to time \( n \).

**Definition.** A random variable \( T \) is called a stopping time with respect to \( \{F_n\}_{n \geq 0} \) if one can answer the question "\( T \geq n? \)" by examining \( F_n \) for all \( n \geq 0 \). Formally, \( \{T \geq n\} \in F_n \).

**Example.** Let \( E = \mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, \ldots\} \). Initially, \( X_0 = 0 \). \( P(X_{n+1} = X_n + 1 \mid X_n) = P(X_{n+1} = X_n - 1 \mid X_n) = \frac{1}{2}, \) \( n \geq 0 \).

This is a Markov chain.

\[ T = \min\{k \geq 1 : X_k = 0\} \] is a stopping time.
Theorem [Strong Markov Property] Given HMC \( \{X_{n}\}_{n=0} \), a stopping \( \tau \) and let \( X_{\tau} = i \) for some \( i \in \mathbb{E} \).

Then (a) \( \{X_{0}, \ldots, X_{\tau-1}\} \) and \( \{X_{\tau+n}\}_{n=1}^{\infty} \) are independent given \( X_{\tau} = i \).

(b) The \( \{X_{\tau+n}\}_{n=1}^{\infty} \) is HMC with same transition matrix \( P \).

Proof (a) We wish to establish the following. For any \( \tau \geq 1 \):

\[
P( (X_0 = i_0, \ldots, X_{\tau-1} = i_{\tau-1}) \mid X_{\tau} = i )
\]

\[
= P( X_0 = i_0, \ldots, X_{\tau-1} = i_{\tau-1}) \cdot P( X_{\tau+\tau} = i, \ldots, X_{\infty} = i )
\]

Equivalently:

\[
P( X_{\tau+\tau} = i_1, \ldots, X_{\tau+n} = i_k \mid X_{\tau} = i, X_0 = i_0, \ldots, X_{\tau-1} = i_{\tau-1} )
\]

We will establish the above equality for \( \tau = 1 \). The general proof is similar (use induction).

(A) \[
P( X_{\tau+\tau} = i_1 \mid X_{\tau} = i, X_0 = i_0, \ldots, X_{\tau-1} = i_{\tau-1} ) = \frac{P( X_{\tau+1} = i_1, X_0 = i_0, \ldots, X_{\tau-1} = i_{\tau-1} )}{P( X_{\tau} = i, X_0 = i_0, \ldots, X_{\tau-1} = i_{\tau-1} )}
\]

where we used notation: \( X_{\tau} = (X_0, \ldots, X_{\tau}) \); \( i_0, \ldots, i_{\tau-1} \).
\[
\text{NUMerator of } (A) = \sum_{n \geq 0} P(z = n, X_{n+1} = j \mid X_n = i, X_0^{\text{nt}} = i_0^{\text{nt}}, z = n) \cdot P(z = n, X_n = i, X_0^{\text{nt}} = i_0^{\text{nt}})
\]

Now, \(\{z = n\} = \{z > n+2\} \cap \{z > n\} \in \mathcal{F}_n\). Therefore, by Markovian property of \(X\), we have

\[
P(C, X_{n+1} = j \mid X_n = i, X_0^{\text{nt}} = i_0^{\text{nt}}, z = n) = P(C, X_{n+1} = j \mid X_n = i) = p_{ij}
\]

Thus, \(\text{NUM of } (A) = p_{ij} \sum_{n \geq 0} P(z = n, X_n = i, X_0^{\text{nt}} = i_0^{\text{nt}})\)

\[
= p_{ij} \cdot P(X_{z+1} = i, X_0^{z+1} = i_0^{z+1})
\]

\[
= p_{ij} \left( \text{DENominator of } (A) \right)
\]

Thus, \( (A) = p_{ij} \).

Similar argument shows that:

\[
P(X_{z+1} = j \mid X_z = i) = p_{ij}.
\]

This completes the proof of (simpler version) (a).
Proof (b) we wish to establish that

\[ P \left( X_{t+1} = i_k \mid X_t = i_0 \right) = \prod_{l=0}^{k-1} P_{i_l i_{l+1}} \]

The proof for \( k=1 \) follows using exactly same argument as above. For \( k \geq 1 \), the proof follows by induction.

**Definition.** Given HMC with transition matrix \( P \), \( P^n \) is the \( n \)-step transition matrix. Specifically,

\[ P^n = [P_{ij}(n)] \quad P_{ij}(n) = \text{prob of visiting } j \text{ in the } n^{th} \text{ step starting from } i. \]

**Definition.** Node \( i \) communicates with \( j \) if there exist \( n_1, n_2 \geq 0 \) s.t. \( P_{ij}(n_1) > 0 \) and \( P_{ji}(n_2) > 0 \). This is denoted as \( i \leftrightarrow j \).

**Definition.** Communication defines equivalence classes of HMC: if \( i \leftrightarrow j \), \( j \leftrightarrow k \), then \( i \leftrightarrow k \); \( i \leftrightarrow i \) (i.e., \( P_{ii}(0) = 1 \)).
Definition. A Markov chain is called **irreducible** if there is only one communication class.

**Lemma.** For an irreducible HMC, there exists a partition of $E$ into disjoint sets $C_0, \ldots, C_{d-1}$ s.t.

for all $k \in \{0, \ldots, d-1\}$, $i \in C_k,$

$$\sum_{j \in C_{k+1}} P_{ij} = 1$$

where $C_d = C_0$ and $d$ is max'l (that is, there is no $d' > d$ for which above property holds).

From now on, we will always assume that Markov chain is **irreducible**.

Definition. An irreducible HMC is called aperiodic if its period $d = 1$. 
**Recurrence**

**Definition.** Let \( T_i = \min \{ k \geq 1 : X_k = i \} \). The \( T_i \) is a stopping time. State \( i \) is called **recurrent** if \( P_i(T_i < \infty) = P(T_i < \infty | X_0 = i) = 1 \).

**Lemma.** Let \( N_i = \sum_{n \geq 1} 1_{\{X_n = i\}} \) be number of times state \( i \) is visited. \( n \geq 1 \)

Then, \( P_i(T_i < \infty) = 1 \iff E_i \left[ N_i \right] = \infty \).

**Proof.** Let \( f_{ii} = P_i(T_i < \infty) \). Let \( o = 0, 1, \ldots \), be times of visits of state \( i \) by Markov chain. Now, suppose \( f_{ii} < 1 \). For \( r \geq 1 \), using Strong Markov Property,

\[
P_i(N_i = r) = P_i(\tau_i < \infty; \tau_2 - \tau_1 < \infty, \ldots, \tau_r - \tau_{r-1} < \infty; \tau_{r+1} - \tau_r = \infty)
\]

\[
= \left[ \prod_{j=1}^{r} P_i(\tau_{j} - \tau_{j-1} < \infty) \right] \cdot P_i(\tau_{r+1} - \tau_r = \infty)
\]

\[
= f_{ii}^r (1 - f_{ii})
\]

Therefore, \( E_i \left[ N_i \right] = \sum_{r \geq 1} r \cdot f_{ii}^r (1 - f_{ii}) \)

\[
= \frac{1}{1 - f_{ii}}.
\]
The above equality suggests that:

\[ P_i(T_i < \infty) < 1 \iff \mu_i < 1 \iff E_i[N_i] < \infty \]

Equivalently:

\[ P_i(T_i < \infty) = 1 \iff E_i[N_i] = \infty. \quad \text{QED}. \]

**Lemma.** For irreducible HMC, if some \( i \in E \) is recurrent, then any other \( j \in E \) is recurrent.

**Proof:** First note the following:

\[
E_i[N_i] = E_i \left[ \sum_{n \geq 1} 1_{\{X_n = i\}} \right] = \sum_{n \geq 1} E_i[1_{\{X_n = i\}}] = \sum_{n \geq 1} p_{ii}(n)
\]

Now suppose \( i \) is recurrent. Then,

\[
E_i[N_i] = \sum_{n \geq 1} P_{ii}(n) = \infty.
\]

Now consider any \( j \). Since HMC is irreducible, there exist \( M_1, M_2 \) s.t. \( P_{ij}(M_1) > 0 \) \( \land \) \( P_{ji}(M_2) > 0 \).

Let, \( c = P_{ij}(M_1) \cdot P_{ji}(M_2) > 0. \)
Now \( P_{ij} (n + M_1 + M_2) \geq P_{ij} (M_2) \cdot P_{ii} (n) \cdot P_{ij} (M_1) \)

\[ = c P_{ii} (n). \]

Therefore, \( \sum_{k \geq 1} P_{ii} (n^{k}) \geq c \sum_{n \geq 0} P_{ii} (n) \)

Therefore, \( E_j [N_j] = \sum_{k \geq 1} P_{ii} (n^{k}) \geq c E_i [N_i] \geq \infty \)

since \( E_i [N_i] \geq \infty \). Thus, \( j \) is recurrent. \( \text{QED.} \)
Invariant Measure

Definition. Let \( x = (x_i)_{i \in E} \) be s.t. \( x_i \subseteq \{0, 1\} \) for all \( i \in E \), and \( x^T = x^TP \); that is
\[
x_i = \sum_{j \in E} x_j P_{ji}.
\]
Then, \( x \) is called an invariant measure.

Lemma[Existence] Given an irreducible recurrent HMM there is at least one invariant measure.

Specifically, consider some \( o \in E \). Define,
\[
x^o_i = \mathbb{P}_o \left[ \sum_{n \geq 1} \mathbb{1}_{\{x_n = i\}} \mathbb{1}_{\{n \leq T_o\}} \right]
\]
with \( T_o = \min \{ k \geq 1 : x_k = o \} \). Then, such an \( x^o = (x^o_i) \) is an invariant measure.

Proof. First, note that
\[
x^o = \mathbb{P}_o \left[ \sum_{n \geq 1} \mathbb{1}_{\{x_n = o^3\}} \mathbb{1}_{\{n \leq T_o\}} \right] = 1
\]
\( \leq \cos x_n = 0 \) only when \( n = T_o \) for any \( n \leq T_o \).

Define \( \phi_i(n) = \mathbb{P}_o (x_1 = o, \ldots, x_{n-1} = o, x_n = i) \); for any \( i \in E \).
Then,
\[
x^o_i = \sum_{n \geq 1} \phi_i(n).
\]
Note that \( \phi_i(1) = \text{Poi} \).
Using MC's property, for \( n \geq 2 \)
\[
\psi^e_i(n) = \sum_{j \neq 0} \psi_j(n-1) P_{ji}.
\]

Summing over \( n \) gives:
\[
x^e_i = \sum_{n \geq 1} \psi_j(n) = \sum_{j \neq 0} \left[ \sum_{n \geq 1} \psi_j(n) P_{ji} \right] + \text{Poi}
\]
\[
= \sum_{j \neq 0} \left[ \sum_{n \geq 1} \psi_j(n) \right] P_{ji} + \text{Poi}
\]
\[
= \sum_{j \neq 0} x^e_j P_{ji} + x^e_0 \text{Poi}.
\]
\[
= \sum_{j \neq 0} x^e_j P_{ji}
\]

Thus, \( x^e \) is an invariant measure as long as we show that \( x^e_i \in (0,\infty) \) for all \( i \in E \).
This is true for \( x^e_0 = 1 \) trivially.

By definition of irreducibility, there exist \( M_i > 0 \)
\( s.t. \ P_{0i}(M_i) > 0 \). That is, \( \psi_i(n_i) > 0 \) for \( \alpha, n_i \leq M_i \).
That is, \( x^e_0 = \sum_{n \geq 1} \psi_i(n) > 0 \).

Thus, all \( x^e_i > 0 \) for \( i \in E \).
Finally, note that

\[ l = x^0 = \sum_{j \in \mathbb{E}} x_j^0 \rho_j(n) \quad \text{for all } n \geq 1. \]

Therefore, if \( x_j^0 = \infty \) then the above cannot hold as \( \sum_{j \in \mathbb{E}} \rho_j(n) > 0 \) by definition of irreducibility. Therefore, \( x_j^0 < \infty \) for all \( j \in \mathbb{E} \).

Thus, we have established that \( x^0 = (x^0_j) \) is an invariant measure for irreducible HMC. \( \quad \text{QED} \)

---

**Property of \( x^0 \)**

\[
\sum_{i \in \mathbb{E}} x_i^0 = \sum_{i \in \mathbb{E}} E_0 \left[ \sum_{n \geq 1} 1 \mathbf{I} x_i(n) \leq 1 \mathbf{I} \rho_i \right] = E_0 \left[ \sum_{n \geq 1} 1 \mathbf{I} \rho_i \right] = E_0 \left[ \sum_{n \geq 1} 1 \mathbf{I} \rho_i \right] = E_0 \left[ T_0 \right].
\]
Lemma [Uniqueness of Inv. Measure]

For an irreducible HMC, let \( x = (x_i) \), \( y = (y_i) \) be two invariant measures. If HMC is recurrent then there exists \( c > 0 \) s.t.

\[
x_i = cy_i \quad \text{for all } i \in \mathcal{E}.
\]

Proof. It is sufficient to show that for any inv. measure \( y = (y_i) \), for all \( i \in \mathcal{E} \)

\[
x_i = \frac{1}{y_o} y_i \quad \text{for some fixed } o \in \mathcal{E}.
\]

For this, define matrix \( Q = [q_{ij}] \) as

\[
a_{ji} = \frac{y_i}{y_j} p_{ij}.
\]

Now \( Q \) is a transition matrix on \( \mathcal{E} \); hence

\[
\sum_i q_{ji} = \sum_i \frac{y_i}{y_j} p_{ij} = \frac{1}{y_j} \left[ \sum_i y_i p_{ij} \right] = \frac{y_j}{y_j} = 1.
\]

Here, we used the fact that \( y \) is an inv. measure.

Inductively, assume \( q_{ji}(n) = \frac{y_i}{y_j} p_{ij}(n) \); for \( n \geq 1 \).

Then,

\[
q_{ji}(n+1) = \sum_k q_{kj} q_{ki}(n) = \sum_k p_{kj}(n) \cdot \frac{y_i}{y_j} \cdot p_{ij}(n) = \frac{y_i}{y_j} \cdot p_{ij}(n+1).
\]
Define \( \gamma_j(n) = \mathbb{P}_Q (X_0 = j ; X_1 \neq 0 ; \ldots ; X_{n-1} \neq 0 ; X_n = 0) \); in words, it is the probability of reaching state 0 for the first time starting at \( j \) w.r.t. HMC governed by \( Q \).

Using similar arguments as above, we obtain

\[
\gamma_i(n+1) = \sum_{j \neq 0} q_{ij} \gamma_j(n)
\]

Replacing \( q_{ij} = \frac{y_j}{y_i} p_{ji} \); \( y_i \gamma_i(n+1) = \sum_{j \neq 0} y_j \gamma_j(n) \) \( p_{ji} \)

Define \( \delta_i(n+1) = y_i \gamma_i(n+1) \) for all \( i \in E ; n \geq 0 \).

Then:

\[
\delta_i(n+1) = \sum_{j \neq 0} \delta_j(n) p_{ji}
\]

Recall this recursion is precisely the same as that for \( \psi_i(n+1) \) in proof of Lemma for Existence.

Specifically,

\[
y_0 \psi_i(n+1) = \sum_{j \neq 0} y_0 \psi_j(n) p_{ji}
\]

Clearly \( y_0 \psi_i(1) = y_0 \cdot p_{i0} = y_i \cdot q_{i0} = y_i \gamma_i(1) = \delta_i(1) \).

Thus, for \( n=1 \) \( \delta_i(n) ; y_0 \psi_i(n) \) are the same. Hence, recursively they are the same.

That is:

\[
y_0 \psi_i(n) = \delta_i(n) = y_i \gamma_i(n)
\]

Now:

\[
\sum_{n \geq 1} \gamma_i(n) = \mathbb{P}_Q \left( \text{reaching state 0 starting from } j \text{ eventually} \right) = 1
\]

Since \( Q \) is recurrent by def. of \( Q \) and recurrence of \( P \) [check this: note \( q_{ii}(n) = \psi_i(n) ; q_{ij}(n) > 0 \iff p_{ij}(n) > 0 \)]
Therefore, \( y_0 \left[ \sum_{n} \varphi_i(n) \right] = y_i \)

But \( \sum_{n} \varphi_i(n) = x_i \) as defined in lemma.

Thus, \( y_i = y_0 \cdot x_i \).

This completes the proof. Q.E.D.
Positive Recurrence

Definition. State $i$ of an HMC is positive recurrent if $E_i[T_i] < \infty$. Clearly, a state is recurrent if it is positive recurrent. But, not otherwise.

Example. 1-D random walk. Details later.

HMC is positive recurrent if all states are positive recurrent.

Lemma. [Alternate definition]

State $0 \in E$ is positive recurrent if and only if

$$\sum_{i \in E} x_i^0 < \infty.$$ 

Proof.

By definition:

$$x_i^0 = E_0 \left[ \sum_{n \geq 1} \mathbb{1}_{\{X_n = i\}} \mathbb{1}_{\{T \leq T_0\}} \right].$$

Therefore,

$$\sum_{i \in E} x_i^0 = E_0 \left[ \sum_{n \geq 1} \mathbb{1}_{\{X_n \leq T_0\}} \left( \sum_{i \in E} \mathbb{1}_{\{X_n = i\}} \right) \right]$$

$$= E_0 \left[ \sum_{n \geq 1} \mathbb{1}_{\{X_n \leq T_0\}} \right] = E_0 \left[ T_0 \right].$$

Thus, $0$ is positive recurrent if and only if $E_0 \left[ T_0 \right] < \infty$.

$$\Leftrightarrow \sum_{i \in E} x_i^0 < \infty. \quad Q.E.D.$$
Lemma [Equivalence of Positive Recurrence]

Given an irreducible HMC, if some OGE is positive recurrent, then all iGE are positive recurrent.

Proof. If OGE is positive recurrent, then it is recurrent. The HMC is irreducible. Hence, by property of recurrence all state iGE are recurrent. Therefore, all invariant distributions of HMC are unique up to multiplicative constant.

Consider any state \( i \in \mathcal{E} \) s.t. \( i \neq 0 \).

As before we can define two invariant measures \( \mathcal{X}^0 \) and \( \mathcal{X}^l \).

Now as discussed above,

\[
\mathcal{X}^l = c \mathcal{X}^0 \quad \text{for some} \quad c > 0.
\]

Further, \( \mathcal{X}^l_i = 1 \). Therefore, \( c = \frac{1}{\mathcal{X}^0_i} \) which is finite. Therefore

\[
\sum_{i \in \mathcal{E}} \mathcal{X}^0_i = c \left( \sum_{i \in \mathcal{E}} \mathcal{X}^0_i \right) = c \cdot E_0[T_0]
\]

\[
= \frac{1}{\mathcal{X}^0_i} \cdot E_0[T_0] < \infty.
\]

That is, \( i \) is positive recurrent by prev. lemma. QED
Lemma. Given an irreducible HMC, it is positive recurrent if $E$ is finite.

Proof. First, we show that it is recurrent.

By definition of recurrence $\sum_{n \geq 1} p_{ij}(n) = \infty$ for all $i \in E$.

For irreducible HMC, this implies that $\sum_{n \geq 1} p_{ij}(n) = \infty$.

$B'cos \sum_{n \geq 1} p_{ij}(n) = E_i \left[ N_j \right] \geq P_i(\text{visit } j \text{ once}) \cdot E_j \left[ N_j \right]$.

and for irreducible recurrent chain $P_i(\text{visit } j \text{ once}) = 1$.

Therefore using fact $E_j \left[ N_j \right] = \infty$ gives $\sum_{n \geq 1} p_{ij}(n) = \infty$.

Suppose, HMC that is on finite state space $E$ and irreducible is not recurrent. Then:

$\sum_{n \geq 1} p_{ij}(n) < \infty$ for all $i,j \in E$.

$\Rightarrow \sum_{n \geq 1} \sum_{j \in E} p_{ij}(n) < \infty$ since $E$ is finite.

$\Rightarrow \sum_{n \geq 1} E_i \left[ \sum_{k \in E} 1 \right] < \infty \Rightarrow \sum_{n \geq 1} 1 < \infty$.

contradiction! Hence, HMC is recurrent.
Given the recurrent HMC, we know that for any state $0 \in \mathcal{E}$, $x^0 = (x_i^0)$ is an invariant measure. That is $x_i^0 \in (0, \infty) + i \in \mathcal{E}$. But $\mathcal{E}$ is finite. Hence

$$\sum_{i \in \mathcal{E}} x_i^0 < \infty$$

since each $x_i^0 < \infty$.

This is the condition characterizing positive recurrence of HMC.

Thus, we have established that any finite irreducible HMC is positive recurrent. \( \square \)
Stationary Distribution: Existence, Uniqueness

Definition. Let \( \{\Pi(i)\} \) be an invariant measure of HMC \( P \) such that \( \sum \Pi(a) = 1 \). Then \( \Pi = [\Pi(a)] \) is called stationary distribution of HMC.

Lemma. For an irreducible positive recurrent HMC, there exists unique stationary distribution.

Proof. The HMC is irreducible and positive recurrent. Hence, given any \( 0 \in E \), \( x^0 = (x^0_i)_{i \in E} \) is an invariant measure with \( \sum x^0_i < \infty \). Then, define \( \Pi(i) = \frac{x^0_i}{\sum_{j \in E} x^0_j} \). The \( \Pi = [\Pi(a)] \) is stationary distribution. If there exists any other stationary distribution \( \hat{\Pi} = [\hat{\Pi}(a)] \), then by recurrence property of HMC; \( \hat{\Pi}(a) \) constant \( c > 0 \) s.t.

\[ \Pi(a) = c \hat{\Pi}(a) \text{ for all } a \in E. \]

But,

\[ 1 = \sum_{i \in E} \Pi(a) = c \sum_{i \in E} \hat{\Pi}(a) = c \cdot \sum_{i \in E} \hat{\Pi}(a) = c. \]

Thus \( \Pi = \hat{\Pi} \). That is, \( \Pi \) is unique such stationary distribution. \( \Box \)
**Stationary Distribution: Convergence**

**Definition.** Given distributions $\mu$ and $\nu$ on $E$, define distance between $\mu$, $\nu$ as

$$d_{TV}(\mu, \nu) = \sup_{A \subseteq E} \{\mu(A) - \nu(A)\}.$$ 

**Lemma.** Given an irreducible, aperiodic and positive recurrent HMC on countable state-space $E$, starting from any distributions $\mu, \nu$ on $E$

$$\lim_{n \to \infty} d_{TV}(\mu^T \rho^n, \nu^T \rho^n) = 0.$$ 

That is,

$$\lim_{n \to \infty} \left| \mu^T \rho^n - \pi \right| = 0.$$
TEST FOR POSITIVE RECURRENCE:

FOSTER'S CRITERIA

Lemma I. Given an irreducible HMC on countable state space \( E \), let there exist a non-negative valued function \( V: E \rightarrow \mathbb{R}_+ \) such that:

(a) \( \sum_{j \in E} P_{ij} V(j) < \infty \) for all \( i \in E \);

(b) \( \sum_{j \in E} P_{ij} V(j) \leq V(i) - \epsilon \), for all \( i \neq f \)

where \( \epsilon > 0 \) and \( F \) a finite subset of \( E \).

Then, HMC is positive recurrent.

Proof. The above Lemma I (Foster's Criteria) is implied by the following two Lemmas, which are proved later.

Lemma II. Under hypothesis of Lemma I, for any \( i \in F \), \( E_i[\tau_F] < \infty \), where \( \tau_F = \min \{ k \geq 1 : X_k \in F \} \).

Lemma III. For an irreducible HMC, if there is a finite set \( F \) s.t. for any \( i \in F \), \( E_i[\tau_F] < \infty \), then HMC is positive recurrent. 

Q.E.D.
Proof \[ \text{[Lemma II]} \]

Let \( X_n \) denote the state of HMC at time \( n \). Initially, \( x_0 = i_0 \in \mathcal{E}. \) Since \( \mathcal{E} \) is finite set, HMC is irreducible the (a) of Lemma I ensures that \( V(x_0) = V(i) < \infty. \)

Let \( B = \left[ \max_{j \in \mathcal{E}} \sum_{k} P_{jk} V(k) \right] + \varepsilon. \)

Now, let \( \mathcal{F}_n \) denote history of HMC up to time \( n. \)

Then,

\[
\mathbb{E} \left[ V(X_{n+1}) \mid \mathcal{F}_n \right] = \mathbb{E} \left[ V(X_{n+1}) \left( \mathbf{1}_{\{X_n \in \mathcal{E}_n^\prime \}} + \mathbf{1}_{\{X_n \notin \mathcal{E}_n^\prime \}} \right) \mid \mathcal{F}_n \right]
\]

\[
= \mathbb{E} \left[ V(X_{n+1}) \mathbf{1}_{\{X_n \in \mathcal{E}_n^\prime \}} \mid \mathcal{F}_n \right] + \mathbb{E} \left[ V(X_{n+1}) \mathbf{1}_{\{X_n \notin \mathcal{E}_n^\prime \}} \mid \mathcal{F}_n \right]
\]

\[
= \mathbf{1}_{\{X_n \in \mathcal{E}_n^\prime \}} \cdot \left[ \sum_k P_{nk} V(k) \right] + \mathbf{1}_{\{X_n \notin \mathcal{E}_n^\prime \}} \left[ \max_{j \in \mathcal{E}} \sum_{k} P_{jk} V(k) \right]
\]

Now,

\[
\mathbf{1}_{\{X_n \notin \mathcal{E}_n^\prime \}} \left[ \sum_k P_{nk} V(k) \right] \leq \left[ \max_{j \in \mathcal{E}} \sum_{k} P_{jk} V(k) \right] \mathbf{1}_{\{X_n \notin \mathcal{E}_n^\prime \}}
\]

\[
= (B - \varepsilon) \cdot \mathbf{1}_{\{X_n \notin \mathcal{E}_n^\prime \}}
\]

Also,

\[
\mathbf{1}_{\{X_n \in \mathcal{E}_n^\prime \}} \left[ \sum_k P_{nk} V(k) \right] \leq \left( V(x_n) - \varepsilon \right) \cdot \mathbf{1}_{\{X_n \in \mathcal{E}_n^\prime \}}
\]

by hypothesis (b) of Lemma I.
Therefore:

\[
\mathbb{E} \left[ V(x_{n+1}) \mid F_n \right] \leq (B - \varepsilon) \mathbb{1}_{\{x_n \in \mathcal{F}\}} + (V(x_n) - \varepsilon) \mathbb{1}_{\{x_n \notin \mathcal{F}\}}
\]

\[
\leq V(x_n) - \varepsilon + B \mathbb{1}_{\{x_n \in \mathcal{F}\}}. \quad -(*)
\]

Define \( T^m = \min \left\{ T(F), m, Z^m \right\} \) where

\[
Z^m = \min \left\{ \eta > 0 : V(x_n) \geq m^2 \right\}.
\]

By definition, \( T^m \leq m \) w.p. 1 and it is a stopping time. That is event \( \{T^m > n\} \in F_n \).

[In words: \( T^m > n \) or \( T^m \leq n \) can be determined by looking at history \( F_n \).]

Multiply \((*)\) on both sides by \( \mathbb{1}_{\{T^m > n\}} \).

\[
\mathbb{E} \left[ V(x_{n+1}) \mid F_n \right] \cdot \mathbb{1}_{\{T^m > n\}} \leq \left[ V(x_n) - \varepsilon + B \mathbb{1}_{\{x_n \in \mathcal{F}\}} \right] \cdot \mathbb{1}_{\{T^m > n\}}
\]

Now, \( \mathbb{E} \left[ V(x_{n+1}) \mid F_n \right] \cdot \mathbb{1}_{\{T^m > n\}} = \mathbb{E} \left[ \mathbb{1}_{\{T^m > n\}} \cdot V(x_{n+1}) \mid F_n \right] \)

because \( \{T^m > n\} \in F_n \).

Also, \( \mathbb{1}_{\{T^m > n\}} \geq \mathbb{1}_{\{T^m > n+1\}} \); \( V \geq 0 \).

Therefore, \( V(x_n) \cdot \mathbb{1}_{\{T^m > n\}} \geq V(x_n) \cdot \mathbb{1}_{\{T^m > n+1\}} \).

That is:

\[
\mathbb{E} \left[ V(x_{n+1}) \cdot \mathbb{1}_{\{T^m > n\}} \mid F_n \right] \geq \mathbb{E} \left[ V(x_{n+1}) \cdot \mathbb{1}_{\{T^m > n+1\}} \mid F_n \right]
\]
Let \( Y^m_n = V(X_n) \mathbb{I}_{\{T^m > n\}} \). Then,

\[
\mathbb{E}[Y^m_n | \mathcal{F}_n] \leq Y^m_n - \varepsilon \mathbb{I}_{\{T^m > n\}} + B \mathbb{I}_{\{T^m \leq n; X_n \in \mathcal{F}\}}
\]

Take expectations on both sides:

\[
\mathbb{E}[Y^m_{n+1}] \leq \mathbb{E}[Y^m_n] - \varepsilon \mathbb{E}[\mathbb{I}_{\{T^m > n\}}] + B \mathbb{E}[\mathbb{I}_{\{T^m \leq n; X_n \in \mathcal{F}\}}]
\]

Summation of the above equation for \( n = 0, \ldots, L \) gives:

\[
\mathbb{E}\left[\sum_{n=1}^{L} Y^m_n \right] \leq \mathbb{E}\left[\sum_{n=0}^{L} Y^m_n \right] - \varepsilon \sum_{\delta n \leq L} \mathbb{P}(T^m > n) + B
\]

Note that \( Y^m_n = 0 \) for \( n \geq m \) since \( T^m \leq m \) by definition.

Setting \( L = m \):

\[
\mathbb{E}\left[\sum_{n=1}^{L} Y^m_n \right] = \mathbb{E}\left[\sum_{n=0}^{L} Y^m_n \right] = \mathbb{E}\left[\sum_{n=0}^{L} Y^m_n \right] - \mathbb{E}[Y^m_0]
\]

And,

\[
\sum_{\delta n \leq L} \mathbb{P}(T^m > n) = \mathbb{E}[T^m]. \quad (\text{since } \mathbb{P}(T^m > m) = 0).
\]

Therefore, we get:

\[
\varepsilon \mathbb{E}[T^m] \leq \mathbb{E}[Y^m_0] + B
\]

\[
\mathbb{E}[Y^m_0] = \mathbb{E}\left[V(X_0) \mathbb{I}_{\{T^m \leq m\}} \right] \leq \mathbb{E}\left[V(X_0)\right] = V(X_0) = V(\delta_0).
\]
Thus, \( \mathbb{E}[T^m] \leq V(c_0) + B < \infty \).

Now \( T^m \uparrow T(F) \) and hence by above and monotone convergence theorem:

\[ \mathbb{E}[T(F)] \leq V(c_0) + B. \]

That is, \( \mathbb{E}[T(F)] \leq \frac{V(c_0) + B}{\varepsilon} < \infty \). \( \square \)

Now proof of Lemma III that implies that \( \mathbb{E}_0[T(F)] < \infty \) for any \( i_0 \in F \), a finite set is sufficient for positive recurrence of HMC.

**Proof [Lemma III]** Consider \( i_0 \in F \). Since, HMC is irreducible, it is sufficient to establish that \( \mathbb{E}_0[T_{i_0}] < \infty \) for establishing positive recurrence property of HMC.

Now, let \( 0 = T_0, T_1, \ldots \) be successive times when \( X_n \in F \). By strong Markov property,

\( Y_k = X_{T_k} \) is a time homogenous Markov chain (HMC), with state space \( F \). Since original HMC is irreducible, \( Y_k \) is irreducible as well.
Thus, $Y_k$ is irreducible HMC over finite state space. Hence, it is positive recurrent. Let $\tilde{T}_e$ be the stopping time indicating return to state $i \in F$ with respect to "visits to $F$". The positive recurrent implies

$$E_\pi(\tilde{T}_e) < \infty \quad \text{for all } \pi \in F.$$ 

Since $\pi$ is finite:

$$\left( \max_{\pi \in F} E_\pi(\tilde{T}_e) \right) < \infty.$$ 

In words: average number of visits it takes the HMC to $F$ in order to revisit any state $i \in F$ again is finite.

We want to use this along with hypothesis of Lemma III that expected return time to $F$ is finite to establish $E_{i_0}(\tilde{T}_{i_0}) < \infty$ for any $i_0 \in F$.

For this, define $S_k = \tilde{T}_{k+1} - \tilde{T}_k$, that is the inter-visit time to $F$ for the $k+1^{st}$ time.
Using above notation, we can re-write: $i_0 \in F$,

$$E_{i_0} \left[ T_{i_0} \right] = E_{i_0} \left[ \sum_{k \geq 0} S_k \frac{1}{2} \mathbb{1}_{T_{i_0} > k} \right]$$

$$= E_{i_0} \left[ \sum_{k \geq 0} S_k \cdot \mathbb{1}_{T_{i_0} > k} \right] \left[ \sum_{l \in F} \mathbb{1}_{\{x_{z_k} = l\}} \right]$$

$$= E_{i_0} \left[ \sum_{l \in F} \sum_{k \geq 0} \mathbb{1}_{\{x_{z_k} = l\} \cdot \mathbb{1}_{T_{i_0} > k} \cdot S_k} \right]$$

$$= \sum_{l \in F} \sum_{k \geq 0} E_{i_0} \left[ S_k \mid x_{z_k} = l ; \tilde{T}_{i_0} > k \right] \cdot P_{i_0} \left( x_{z_k} = l ; \tilde{T}_{i_0} > k \right)$$

Now: $E_{i_0} \left[ S_k \mid x_{z_k} = l ; \tilde{T}_{i_0} > k \right] = E_{i_0} \left[ z_{k+1} - z_k \mid x_{z_k} = l \right]$

$$= E_{\mathbb{E}} \left[ T(F) \right],$$

where we used facts: $\{\tilde{T}_{i_0} > k\} \in F_{z_k}$, strong Markov property of HMC. Therefore,

$$E_{i_0} \left[ T_{i_0} \right] = \sum_{l \in F} \sum_{k \geq 0} E_{\mathbb{E}} \left( T(F) \right) \cdot P_{i_0} \left( x_{z_k} = l ; \tilde{T}_{i_0} > k \right)$$

$$\leq \left( \max_{\mathbb{E}} E_{\mathbb{E}} \left( T(F) \right) \right) \sum_{k \geq 0} \left[ \sum_{l \in F} P_{i_0} \left( x_{z_k} = l ; \tilde{T}_{i_0} > k \right) \right]$$

$$= \left( \max_{\mathbb{E}} E_{\mathbb{E}} \left( T(F) \right) \right) \left[ \sum_{k \geq 0} P_{i_0} \left( \tilde{T}_{i_0} > k \right) \right]$$

$$= \left( \max_{\mathbb{E}} E_{\mathbb{E}} \left( T(F) \right) \right) E_{i_0} \left[ \tilde{T}_{i_0} \right].$$
But \((\max_{e} E_e(T(e))) < \infty\) as explained above from hypothesis of Lemma III and 
\(E_{i_0}[\widetilde{T}_{i_0}] < \infty\) as explained using positive recurr. 
of the chain \(\gamma_k\).

Thus: \(E_{i_0}[T_{i_0}] < \infty\) for any \(i_0 \in EF\).

That is, HMC is positive recurrent \(\Box\)
IMPLICATIONS

- Positive recurrence implies existence of stationary distribution.
  - Suppose $E = \mathbb{N} = \{0, 1, 2, \ldots \}$.
  - $\Pi = (\pi(i))_{i \in E}$ be stationary distribution.
  - That is, $\sum_{i \in \mathbb{N}} \pi(i) = 1$.
  - Hence, $\frac{\mathbb{P}(\{n, \infty\})}{\pi(i)} = \sum_{i \geq n} \pi(i) \rightarrow 0$.
  - That is, with respect to $\Pi$ the value of random variable $X_{MC}$ is finite with prob 1.

- The aperiodicity establishes that positive recurrent irreducible HMC converges to stationary distribution.

- Thus, in "equilibrium" an aperiodic, irreducible positive recurrent HMC is finite with prob 1.
Plan:

(I) Check for irreducibility.

(II) Add self-loops to each state with probability $\frac{1}{2}$ (if required).
   - This does not change stationary distribution.
   - Makes HMC aperiodic.

(III) Check for positive recurrence
   - Use of Foster's Criteria.