1. (25 pts) This problem illustrates conditions under which the Poisson approximation is valid.

Let $n$ (very large) sensor nodes, numbered $0, 1, \ldots, n-1$, be thrown in a geographic area to measure temperature $T$. Let $X_i$ denote the output of the $i^{th}$ sensor, where $X_i$ is a binary random variable such that $\Pr[X_i = 1] = T/n$. Let the cumulative sensed temperature be $S_n = \sum_{i=1}^{n} X_i$. Under which of the following conditions do you think $S_n \approx \text{Poi}(T)$? Here $\text{Poi}(T)$ is a Poisson random variable with mean $T$.

(i) The $X_i$s are independent of each other.

(ii) For any $i$, $X_i$ depends on $X_{(i+k \bmod n)}$ for $k \leq \log n$; $X_i$ is independent of all other $X_j$s. Further, $\mathbb{E}[X_i X_{(i+k \bmod n)}] = \frac{T}{n}$ for $k \leq \log n$.

(iii) For any $i$, $\mathbb{E}[X_i X_{(i+k \bmod n)}] = \frac{T}{n} \alpha^k$ with $\alpha < 1$.

2. (25 pts) This problem requires qualitative analysis of maximum load for load balancing policy. Consider the standard ball-bin model: $n$ balls need to be thrown into $n$ bins. The balls are thrown into bins one-by-one. Each ball chooses two bins as follows: first bin is chosen uniformly at random from $n$ bins; the second bin is chosen by choosing the bin on the left or right of the first bin with probability 1/2 each. The ball joins the bin with fewest balls (ties broken arbitrarily). Answer the following questions:

(i) Is the maximum load under the above described policy smaller than the maximum load under the join a bin at random policy (as described in class)? If yes, in what sense is it smaller? If no, why?

(ii) Qualitatively, under the above described policy, does the maximum load scale like $\log n$ or $\log \log n$?

3. (50 pts) This problem is about dual model of the super-market model. Consider $n$ queues with jobs arriving to them according to independent Poisson processes of rates $\lambda_1, \ldots, \lambda_n$ respectively. There is a common Exponential server of rate $n$. When server becomes empty, it chooses one of the $n$ queues to serve according the $d$-policy: choose $d$ queues at random and serve the queue with maximum number of jobs. Answer the following questions.

(i) For $\lambda_1 = \cdots = \lambda_n = \lambda < 1$, does there exist unique stationary distribution so that the net queue-size is finite with probability 1 for any $d \geq 1$? Provide complete justification for your answer.

(ii) Under the setup of (i), consider limiting system as $n \to \infty$. For such system, describe appropriate mean-field model. Do not worry about justification for the model.

(iii) Obtain the fixed point for mean field model for $d = 1$.

(iv) Now, consider the fixed point for mean field model for $d \geq 2$. Similar to super-market model, is there any significant difference in queue-sizes between $d = 1$ and $d \geq 2$ policies?