1. (50 pts) This problem is about establishing lower bound on computation time of distributed algorithms.

Consider an arbitrary connected network of $n$ nodes. Let one of the $n$ nodes has an important piece of information that it needs to send to all nodes. The communication can happen between two nodes if they are connected. The time is slotted. In the beginning of each time-slot, a node can decide to exchange information (or communicate) with at most one of it’s neighbor. The information is small enough so that all of it can be transferred in a time-slot. Answer the following questions:

(a) What is the minimum number of time-slots required for all nodes to have the information for any graph and any algorithm in the above communication model?

(b) Let the bound obtain in (a) be $B_n$ for a graph with $n$ nodes. Is there a graph for which a simple randomized algorithm can spread information to all nodes in $O(B_n)$ time-slots on average? Provide complete justification for your answer.

(c) Can you use answer of (a) to obtain lower bound of $O(\log \frac{1}{\epsilon})$ on $\epsilon$-averaging time $T_\epsilon$, under the above communication model? Recall that in the class, we proved this lower-bound on algorithms based on pair-wise averaging; here we are asking for a lot stronger lower bound. Also, recall that $T_\epsilon$ was defined as follows. Let $x(0) = [x_i]$ be vector of initial values at $n$ nodes of the graph. Goal is to compute the average value: $x_{ave} = \frac{1}{n} \sum_{i=1}^{n} x_i$. Let $x(t)$ be the vector of values at nodes at the end of time-slot $t$ under an averaging algorithm. Then, the $\epsilon$-averaging time of this algorithm, denoted by $T_\epsilon$ is defined as follows:

$$T_\epsilon = \sup_{\|x(0)\| = 1} \inf \{ t : \Pr[\|x(t) - x_{ave}\| \leq \epsilon] \geq 1 - \epsilon \},$$

where $\|x\| = \sum_{i=1}^{n} x_i^2$.

2. (50 pts) This problem is about algorithm design for max-min fair allocation.

First, consider $n$ flows with demands $f_1, \ldots, f_n$ passing through a single-link of capacity $C$. We have defined the max-min fair allocation in the class. Answer the following question:

(a) Is there a rate-allocation that is max-min fair for any set of demands $f_1, \ldots, f_n$ and capacity $C$? Provide an algorithm to find max-min allocation if it exists. Is it unique?

Now, as stated in the class consider the situation of a network, with network graph $G = (V, E)$ and pre-determined routing. In this case, there are $n$ flows with routing matrix $M$ ($M_{ie} = 1$ if flow $i$ passes through link $e \in E$ and 0 otherwise.) The notion of max-min fair was defined in the class for this setup as well. Now, answer the similar question:

(b) Is there a rate-allocation that is max-min fair for any set of demands $f_1, \ldots, f_n$ and link-capacities $C_e, e \in E$? Provide an algorithm to find max-min allocation if it exists. Is it unique?