Convolution Property Example

**Example 5.13** \( h[n] = \alpha^n u[n] \), \( x[n] = \beta^n u[n] \), \( |\alpha|, |\beta| < 1 \)

(O & W p385)

\[
H(e^{j\omega}) = \frac{1}{1 - \alpha e^{-j\omega}}, \quad X(e^{j\omega}) = \frac{1}{1 - \beta e^{-j\omega}}
\]

\[
y[n] = h[n] * x[n] \quad \text{\( \longleftrightarrow \text{ Y}(e^{j\omega}) = \frac{1}{1 - \alpha e^{-j\omega}} \cdot \frac{1}{1 - \beta e^{-j\omega}} \)}
\]

- ratio of polynomials in \( \nu = e^{-j\omega} \)

\( \beta \neq \alpha : \)

\[
Y(e^{j\omega}) = \text{PFE} \frac{A}{1 - \alpha e^{-j\omega}} + \frac{B}{1 - \beta e^{-j\omega}}
\]

A, B \(-\) determined by partial fraction expansion

\[
y[n] = A\alpha^n u[n] + B\beta^n u[n].
\]

\( \beta = \alpha : \)

\[
Y(e^{j\omega}) = \frac{1}{(1 - \alpha e^{-j\omega})^2}
\]

\[
y[n] = \int_{-\infty}^{\infty} y[n] \leftrightarrow j \frac{dY(e^{j\omega})}{d\omega}
\]

\[
y[n] = (n + 1)\alpha^n u[n]
\]
DT LTI System Described by LCCDE’s

\[ \sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k] \]

From time-shifting property:

\[ x[n-k] \leftrightarrow e^{-j\omega k} X(e^{j\omega}) \]

\[ \mathcal{F}\mathcal{T} \text{ on both sides: } \sum_{k=0}^{N} a_k e^{-j\omega k} Y(e^{j\omega}) = \sum_{k=0}^{M} b_k e^{-j\omega k} X(e^{j\omega}) \]

Devide both sides by \( \sum_{k=0}^{N} a_k e^{-j\omega k} \):

\[ Y(e^{j\omega}) = \left[ \frac{\sum_{k=0}^{M} b_k e^{-j\omega k}}{\sum_{k=0}^{N} a_k e^{-j\omega k}} \right] X(e^{j\omega}) \]

— Rational function of \( e^{j\omega} \), use PFE to get \( h[n] \).

\[ h[n] = \alpha^n u[n] \quad \text{— infinite duration} \]

Example:  First-order recursive system

\[ y[n] - \alpha y[n-1] = x[n], \quad |\alpha| < 1 \]

with the condition of initial rest \( \Rightarrow \) causal

\[ \mathcal{F}\mathcal{T} \text{ on both sides: } (1 - \alpha e^{-j\omega}) Y(e^{j\omega}) = X(e^{j\omega}) \]

\[ H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{1}{1 - \alpha e^{-j\omega}} \]

\[ h[n] = \alpha^n u[n] \quad \text{— infinite duration} \]
**DTFT Multiplication Property**

\[ y[n] = x_1[n] \cdot x_2[n] \quad \longleftrightarrow \quad Y(e^{j\omega}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta = \frac{1}{2\pi} X_1(e^{j\theta}) \otimes X_2(e^{j\omega}) \quad \text{periodic convolution} \]

**Proof:**

\[ Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x_1[n] \cdot x_2[n]e^{-j\omega n} \]

Use synthesis Eq. for \( x_1[n] \):

\[ \sum_{n=-\infty}^{\infty} \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(e^{j\theta})e^{j\theta n}d\theta \right)x_2[n]e^{-j\omega n} \]

Exchange order of \( \sum \) and \( \int \):

\[ \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{n=-\infty}^{\infty} X_1(e^{j\theta})x_2[n]e^{-j(\omega-\theta)n}d\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} X_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta \quad \text{QED} \]

**Example:**

\[ y[n] = \left[ \frac{\sin(\pi n/4)}{\pi n} \right]^2 = x_1[n] \cdot x_2[n] \quad , \quad x_1[n] = x_2[n] = \frac{\sin(\pi n/4)}{\pi n} \]

\[ Y(e^{j\omega}) = \frac{1}{2\pi} X_1(e^{j\omega}) \otimes X_2(e^{j\omega}) \]

\[ X_1(e^{j\theta}) \]

\[ X_2(e^{j(\omega-\theta)}) \]

\[ Y(e^{j\theta}) \]
Duality in Fourier Analysis
Fourier Transform is highly symmetric

CTFT: Both time and frequency are continuous and in general aperiodic

\[ x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega \]
\[ X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt \]

Suppose \( f(*) \) and \( g(*) \) are two functions such that they are a Fourier-transform pair (Eg. \( f \) is a square and \( g \) is a sinc),

\[ x_1(t) = f(t) \leftrightarrow X_1(j\omega) = g(\omega). \]

Then for another time-domain function such that,

\[ x_2(t) = g(t) \]

then without doing any math, we should know that

\[ X_2(j\omega) = 2\pi f(-\omega) \]

Example of CTFT duality
Square pulse in either time or frequency domain
Duality between CTFS and DTFT

**CTFS**  
\[ x(t) = \sum_{k=\infty}^{+\infty} a_k e^{j\omega_0 t} = x(t + T) \]  
\[ \omega_0 = \frac{2\pi}{T} \]  
\[ a_k = \frac{1}{T} \int_T x(t) e^{-j\omega_0 t} dt \]

*Periodic in time \(\longleftrightarrow\) discrete in frequency*

**DTFT**  
\[ x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega \]  
\[ X(e^{j\omega}) = \sum_{n=\infty}^{+\infty} x[n] e^{-j\omega n} = X(e^{j(\omega + 2\pi)}) \]

*Discrete in time \(\longleftrightarrow\) periodic in frequency*

**Illustration:** Periodic in time \(\longleftrightarrow\) discrete in frequency

It is clear that  
\[ x(t) = \tilde{x}(t) * \sum_{n=\infty}^{\infty} \delta(t - nT) \]  
— sampling function

*Any* periodic signal  
\[ x(t) \]  
is a convolution of \( x(t) \)  
in one period with  
\[ \sum_{n=\infty}^{\infty} \delta(t - nT) \]
**Illustration** (continued):

\[
\tilde{X}(j\omega) = \tilde{X}(j\omega) \times \sum_{k=-\infty}^{\infty} \frac{2\pi}{T} \delta(\omega - \frac{k2\pi}{T})
\]

Also discrete

\[
\sum_{k=-\infty}^{\infty} \frac{2\pi}{T} \delta(\omega - \frac{k2\pi}{T})
\]

\[\cdots \uparrow \quad \uparrow \quad \frac{2\pi}{T} \quad \uparrow \quad \uparrow \quad \cdots\]

\[\frac{-4\pi}{T} \quad \frac{-2\pi}{T} \quad 0 \quad \frac{2\pi}{T} \quad \frac{4\pi}{T} \quad \omega\]

From the symmetry of Fourier Transform, the reverse is also true:

**Discrete in time** \(\xrightarrow{\text{DTFT}}\) **periodic in frequency**

**Demo:** diffraction grating.

---

**DTFS**

**Discrete & periodic in time** \(\longleftrightarrow\) **periodic & discrete in frequency**

\[
x[n] = \sum_{k=-N}^{N} a_k e^{j\omega_0 n} = x[n + N]
\]

\[
\omega_o = \frac{2\pi}{N}
\]

\[
a_k = \frac{1}{N} \sum_{k=-N}^{N} x[n] e^{-j\omega_0 n} = a_{k+N}
\]

\[\downarrow\]

Suppose \(f[*]\) and \(g[*]\) are two functions such that they are a FT pair,

\[
x_1[n] = f[n] \iff a_k = g[k].
\]

Then for another time-domain function such that,

\[
b_k \iff x_2[n] = g[n]
\]

then without doing any math, we should know that

\[
b_k = Nf[-k]
\]
Magnitude and Phase of FT, and Parseval Relation

CT:

\[ x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega \]

\[ X(j\omega) = |X(j\omega)| e^{j\angle X(j\omega)} \]

Parseval Relation:

\[ \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} \frac{1}{2\pi} |X(j\omega)|^2 d\omega \]

Energy distribution in \( \omega \)

DT:

\[ x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega \]

\[ X(e^{j\omega}) = |X(e^{j\omega})| e^{j\angle X(e^{j\omega})} \]

Parseval Relation:

\[ \sum_{n=-\infty}^{\infty} |x[n]|^2 = \int_{-\pi}^{\pi} \frac{1}{2\pi} |X(e^{j\omega})|^2 d\omega \]

Effects of Phase

- **Not** on signal energy distribution as a function of frequency

- **Can** have dramatic effect on signal shape/character
  - Constructive/Destructive interference

- Is that important?
  - Depends on the signal and the context, not as important as the amplitude in hearing, but **very important** for image processing.
**Demo:**  
1) Effect of phase on Fourier Series

![Graphs of three sinusoids with varying phase](image1)

**Demo:**  
2) Effect of phase on image processing

![ original image of a boat](image2)  
Magnitude of $\mathcal{F}T$  
Phase of $\mathcal{F}T$
Demo: 2) Effect of phase on image processing (cont.)

Construct with the magnitude of $\mathcal{F}T$ of the boat, but with a uniform phase. Construct with the phase of $\mathcal{F}T$ of the boat, but with a uniform magnitude.

Demo: 2) Effect of phase on image processing (cont.)

This image was constructed with the phase of $\mathcal{F}T$ of the boat, but with magnitude from a very different image — you wouldn’t have guess it.
**Demo:** 2) Effect of phase on image processing (cont.)

The magnitude information for the previous image is from our old friend! Cannot make a monkey out of a boat!

Next lecture covers O & W pp. 427-472.