Properties of z-Transforms

\[ x[n] \stackrel{Z}{\longleftrightarrow} X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \]

(1) Time Shifting

\[ x[n-n_0] \leftrightarrow z^{-n_0} X(z) \]

The rationality of \( X(z) \) is unchanged, different from \( \mathcal{L}T \). ROC unchanged except for the possible addition or deletion of the origin or infinity

\( n_0 > 0 \Rightarrow \text{ROC } z \neq 0 \text{ (maybe)} \)
\( n_0 < 0 \Rightarrow \text{ROC } z \neq \infty \text{ (maybe)} \)

(2) \( z \)-Domain Differentiation

\[ nx[n] \leftrightarrow -z \frac{dX(z)}{dz} \]

Derivation

\[ X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \]

\[ \frac{dX(z)}{dz} = -\sum_{n=-\infty}^{\infty} nx[n]z^{-n-1} \]

\[-z \frac{dX(z)}{dz} = \sum_{n=-\infty}^{\infty} nx[n]z^{-n} \quad \text{QED} \]
Convolution Property and System Functions

\[ x[n] \rightarrow h[n] \rightarrow y[n] = x[n] * h[n] \]

\( Y(z) = H(z)X(z) \), ROC at least the intersection of the ROC’s of \( H(z) \) and \( X(z) \), can be bigger if there are pole/zero cancellation. \textit{e.g.}

\[
H(z) = \frac{1}{z - a} \quad |z| > a
\]

\[
X(z) = z - a \quad z \neq \infty
\]

\[ \Rightarrow Y(z) = 1 \quad \text{ROC all } z \]

\[ H(z) = \sum_{n=-\infty}^{\infty} h[n]z^{-n} \quad \text{— The System Function} \]

CAUSALITY

(1) \( h[n] \) right-sided \( \Rightarrow \) ROC is the exterior of a circle \textit{possibly} including \( z = \infty \):

\[
H(z) = \sum_{n-N_{1}}^{\infty} h[n]z^{-n}
\]

If \( N_{1} < 0 \), then the term \( h[N_{1}]z^{-N_{1}} \rightarrow \infty \) at \( z = \infty \)

\[ \Rightarrow \text{ROC outside a circle, but does not include } \infty. \]

Causal \( \iff N_{1} \geq 0 \)

\[ \downarrow \]

\text{A DT LTI system with system function } H(z) \text{ is causal } \iff \text{the ROC of } H(z) \text{ is the exterior of a circle including } z = \infty \]
DT LTI Systems Described by LCCDE’s

\[ \sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k] \]

Use the time-shift property

\[ \sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z) \]

\[ \Downarrow \]

\[ Y(z) = H(z) X(z) \]

\[ H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} \quad \text{— Rational} \]

ROC: Depends on Boundary Conditions, left-, right-, or two-sided. For Causal Systems \( \Rightarrow \) ROC is outside the outermost pole.

Causality for Systems with Rational System Functions

\[ H(z) = \frac{b_M z^M + b_{M-1} z^{M-1} + \cdots + b_1 z + b_0}{a_N z^N + a_{N-1} z^{N-1} + \cdots + a_1 z + a_0} \]

\[ \Downarrow \text{ No poles at } \infty, \text{ if } M \leq N \]

A DT LTI system with rational system function \( H(z) \) is causal \( \Leftrightarrow \) the ROC is the exterior of a circle outside the outermost pole, including \( \infty \). Thus

\[ N \geq M \]
Stability

- LTI System Stable ⇔ \( \sum_{n=-\infty}^{+\infty} |h[n]| < \infty \) ⇔
  
  ROC of \( H(z) \) includes the unit circle \(|z| = 1\)

  \[ \Rightarrow \text{Frequency Response } H(e^{j\omega}) \text{ (DTFT of } h[n] \text{) exists.} \]

- A causal LTI system with rational system function is stable ⇔ all poles are inside the unit circle, i.e. have magnitudes < 1.

Geometric Evaluation of a Rational \( z \)-Transform

**Example #1:**
\( X_1(z) = z - a \) — A first-order zero

**Example #2**
\( X_2(z) = 1/(z - a) \) — A first-order pole

\[ |X_2(z)| = 1/|X_1(z)|, \quad \angle X_2(z) = -\angle X_1(z) \]

**Example #3:**
\( X(z) = M \prod_{i=1}^{K}(\bar{z} - \bar{\beta_i}) \prod_{j=1}^{P}(z - \alpha_j) \)

\[ |X(z)| = |M| \prod_{i=1}^{K} |\bar{z} - \bar{\beta_i}| \prod_{j=1}^{P} |z - \alpha_j| \]

\[ \angle X(z) = \angle M + \sum_{i=1}^{K} \angle (\bar{z} - \bar{\beta_i}) - \sum_{j=1}^{P} \angle (z - \alpha_j) \]
Geometric Evaluation of DT Frequency Responses

First-Order System
— one real pole

\[ H(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z - a} \quad , \quad |\alpha| > |\alpha| \]

\[ h[n] = a^n u[n] \quad , \quad |\alpha| < 1 \]

\[ H(e^{j\omega}) = \frac{v_1}{v_2} \quad , \quad |H(e^{j\omega})| = \frac{|v_1|}{|v_2|} = \frac{1}{|v_2|}, \quad \angle H(e^{j\omega}) = \angle v_1 - \angle v_2 = \omega - \angle v_2 \]

Second-Order System

Two poles that are a complex conjugate pair \((z_1 = re^{j\theta} = z_2^*)\)

\[ H(z) = \frac{z^2}{(z - z_1)(z - z_2)} = \frac{1}{1 - (2r \cos \theta)z^{-1} + r^2z^{-2}}, \quad 0 < r < 1, \quad 0 \leq \theta \leq \pi \]

\[ |H(e^{j\omega})| = \frac{1}{|e^{j\omega} - re^{j\theta})(e^{j\omega} - re^{-j\theta})|}, \quad h[n] = r^n \frac{\sin[(n + 1)\theta]}{\sin \theta} u[n] \]

Clearly, \(|H|\) peaks at \(\omega = \pm \theta\) (remember the tent analogy).
**Demo:** DT pole-zero diagrams, frequency response, vector diagrams, and impulse- & step-responses

![Pole-zero diagrams](image1)

![Frequency response](image2)

**System Function Algebra and Block Diagrams**

Feedback System
(Causal systems)

\[ H(z) = \frac{Y(z)}{X(z)} = \frac{H_1(z)}{1 + H_1(z)H_2(z)} \]

**Black’s formula:**

**Example:**

\[ H(z) = \frac{1}{1 - \frac{1}{4}z^{-1}} \]

\[ y[n] = \frac{1}{4}y[n-1] + x[n] \]

A delay in the feedback branch results in an exponential \( h[n] \) — echo.
**Unilateral z-Transform** – Many Parallels with **ULT**

\[ \mathcal{X}(z) = \sum_{n=0}^{+\infty} x[n]z^{-n} \]

Note:

1. If \( x[n] = 0 \) for \( n < 0 \), then \( \mathcal{X}(z) = X(z) \)

2. **UZT** of \( x[n] = BZT \) of \( x[n]u[n] \)
   \[ \Rightarrow \text{ROC always outside a circle and includes } z = \infty \]

3. For causal LTI systems
   \[ \mathcal{X}(z) = H(z) \]

**Properties of Unilateral z-Transform**

Many properties are the same with \( BZT \) *E.g.*

- Convolution property (for \( x_1[n<0] = x_2[n<0] = 0 \))
  \[ x_1[n] \ast x_2[n] \xrightarrow{UZ} \mathcal{X}_1(z) \cdot \mathcal{X}_2(z) \]

- But there are important differences. For example, *time-shift*
  \[ y[n] = x[n-1] \xrightarrow{UZ} \mathcal{Y}(z) = x[-1] + z^{-1} \mathcal{X}(z) \]

Derivation:

\[ \mathcal{Y}(z) = \sum_{n=0}^{+\infty} y[n]z^{-n} = \sum_{n=0}^{+\infty} x[n-1]z^{-n} = x[-1] + \sum_{n=1}^{+\infty} x[n-1]z^{-n} \]

\[ = x[-1] + z^{-1} \sum_{n=1}^{+\infty} x[n-1]z^{-(n-1)} \]

\[ \xrightarrow{\text{QED}} \mathcal{X}(z) \]
Use of $\mathcal{UZT}'s$ in Solving Difference Equations with Initial Conditions

\[ y[n] + 2y[n-1] = x[n] \]

\[ y[-1] = \beta, \ x[n] = \alpha u[n] \leftrightarrow \frac{\alpha}{1-z^{-1}} \]

$\mathcal{UZT}$ of Difference Equation

\[
\mathcal{UZT}(y[n]) \quad \mathcal{UZT}(x[n]) = \frac{\alpha}{1-z^{-1}}
\]

\[ \mathcal{UZT}(y[n]) + 2\left[ \beta + z^{-1}\mathcal{UZT}(y[n]) \right] = \frac{\alpha}{1-z^{-1}} \]

\[
\mathcal{UZT}(y[n]) = \frac{-2\beta}{1+2z^{-1}} + \frac{\alpha}{(1+2z^{-1})(1-z^{-1})}
\]

$ZIR$ – Output purely due to the initial conditions,  
$ZSR$ – Output purely due to the input.

Example (continued)

\[ \beta = 0 \quad \Rightarrow \quad \text{System is initially at rest:} \]

\[ ZSR \quad \mathcal{U}(z) = \mathcal{H}(z)\mathcal{X}(z) = \frac{1}{1+2z^{-1}} \cdot \frac{\alpha}{1-z^{-1}} \]

\[ \mathcal{X}(z) = H(z) = \frac{1}{1+2z^{-1}} \]

\[ \alpha = 0 \quad \Rightarrow \quad \text{Get response to initial conditions} \]

\[ ZIR \quad \mathcal{U}(z) = -\frac{2\beta}{1+2z^{-1}} \]

\[ \Rightarrow \quad y[n] = -2\beta(-2)^n u[n] \]
Next lecture:

Music CD recording and playback, not in the book. Helpful to review slides 16-20 in Lecture #16 on DT downsampling and upsampling.