Exploiting Superposition and Time-Invariance

\[ x[n] = \sum_k a_k x_k[n] \quad \xrightarrow{\text{Linear System}} \quad y[n] = \sum_k a_k y_k[n] \]

Question: Are there sets of “basic” signals \( x_i[n] \) such that
a) We can represent any signals as linear combinations of these building block signals.
b) The response of LTI Systems to these basic signals are both simple and insightful.

Fact: For LTI Systems (CT or DT) there are two natural choices for these building blocks

Focus for now: DT Shifted unit samples \( \delta[n-n_o] \)
Next time: CT Shifted unit impulses \( \delta(t-t_o) \)
That is ...

\[ x[n] = \ldots + x[-2] \delta[n + 2] + x[-1] \delta[n + 1] + x[0] \delta[n] + x[1] \delta[n - 1] + \ldots \]

Important to note the "-" sign

\[ x[n] = \sum_{k = -\infty}^{+\infty} x[k] \delta[n - k] \]

The Sifting Property of the Unit Sample
Response of DT LTI Systems

\[ x[n] \rightarrow h[n] \rightarrow y[n] \]

- Now suppose the system is LTI, and define the unit sample response \( h[n] \):

\[ \delta[n] \rightarrow h[n] \]

From Time-Invariance:

\[ \delta[n - k] \rightarrow h[n - k] \]

From Linearity:

\[
x[n] = \sum_{k=-\infty}^{+\infty} x[k] \delta[n - k] \rightarrow y[n] = \sum_{k=-\infty}^{+\infty} x[k] h[n - k] = x[n] * h[n]
\]

Hence a Very Important Property of LTI Systems:

The output of any DT LTI System is a convolution of the input signal with the unit-sample response, i.e.

\[
\text{Any DT LTI} \quad \longleftrightarrow \quad y[n] = x[n] * h[n]
\]

\[
= \sum_{k=-\infty}^{+\infty} x[k] h[n - k]
\]

As a result, any DT LTI Systems are completely characterized by its unit sample response.
Graphic View of the Convolution Sum Response of DT LTI systems

Convolution operation procedure

\[ y[n] = x[n] * h[n] = \sum_{k=-\infty}^{+\infty} x[k] h[n-k] \]

- \( h[k] \) \( \xrightarrow{Flip} \) \( h[-k] \)
- \( h[k] \) \( \xrightarrow{Shift} \) \( h[n-k] \)
- \( h[n-k] \) \( \xrightarrow{Multiply} \) \( x[k]h[n-k] \)
- \( x[k]h[n-k] \) \( \xrightarrow{Sum} \) \( \sum_{k=-\infty}^{+\infty} x[k] h[n-k] \)
Visualizing the calculation of $y[n] = x[n] * h[n]$ 

Choose the value of $n$ and consider it fixed 

$$y[n] = \sum_{k=-\infty}^{\infty} x[k] h[n - k]$$

View as functions of $k$ with $n$ fixed

From $x[n]$ and $h[n]$ to $x[k]$ and $h[n-k]$ 

Note, $h[n-k] - k$ is the mirror image of $h[n] - n$ with the origin shifted to $n$

Calculating Successive Values: Shift, Multiply, Sum

$$y[n] = \sum_{k=-\infty}^{\infty} x[k] h[n - k]$$

$y[n] = 0$ for $n <$

$y[-1] =$

$y[0] =$

$y[1] =$

$y[2] =$

$y[3] =$

$y[4] =$

$y[n] = 0$ for $n >$
Examples of Convolution and DT LTI Systems

Ex. #1: \( h[n] = \delta[n] \)

\[
y[n] = x[n] \ast \delta[n] = \sum_{k=-\infty}^{\infty} x[k] \delta[n - k] \\
= x[n] \quad \text{— An Identity system}
\]

Ex. #2: \( h[n] = \delta[n - n_o] \)

\[
y[n] = x[n] \ast \delta[n - n_o] = \sum_{k=-\infty}^{\infty} x[k] \delta[n - n_o - k] \\
= x[n - n_o] \quad \text{— A Shift}
\]

Ex. #3 \( y[n] = \sum_{k=-\infty}^{n} x[k] \quad \text{— An accumulator} \)

Unit Sample response \( h[n] = \sum_{k=-\infty}^{n} \delta[k] = u[n] \)

\[\downarrow\]

therefore: \( x[n] \ast u[n] = \sum_{k=-\infty}^{n} x[k] \)

\[
\begin{array}{c}
u[n] \\
\bullet \bullet \bullet \\
-3 -2 -1 0 1 2 3 \quad n
\end{array}
\]
The Commutative Property of Convolution

\[ y[n] = x[n] * h[n] = h[n] * x[n] \]

Example: Step response \( s[n] \) of an LTI system

\[ s[n] = u[n] * h[n] = h[n] * u[n] \]

"Input" Unit Sample response of accumulator

\[ s[n] = \sum_{k=-\infty}^{n} h[k] \]
The Distributive Property of Convolution

\[ x[n] \ast \{h_1[n] + h_2[n]\} = x[n] \ast h_1[n] + x[n] \ast h_2[n] \]

Interpretation

\[ x[n] \rightarrow h_1[n] + h_2[n] \rightarrow y[n] \]

The Associative Property of Convolution

\[ x[n] \ast (h_1[n] \ast h_2[n]) = (x[n] \ast h_1[n]) \ast h_2[n] \]

(Commutativity)

\[ x[n] \ast (h_2[n] \ast h_1[n]) = (x[n] \ast h_2[n]) \ast h_1[n] \]

Implication (Very special to LTI Systems)
Properties of Convolution

Combining the Commutative property,  
\[ y[n] = x[n] * h[n] = h[n] * x[n] \]

Distributive property,  
\[ x[n] * \{h_1[n] + h_2[n]\} = x[n] * h_1[n] + x[n] * h_2[n] \]

and Associative property,  
\[ x[n] * (h_1[n] * h_2[n]) = (x[n] * h_1[n]) * h_2[n] \]

symbolically, we can treat “∗” as a “×”. Easy, piece of cake! The hard part is the actual calculation of the convolution.

Flip → Shift → Multiply → Sum.

Soon we will develop a clever way (transform) to perform “×” instead of “∗” operation.

Some Useful Properties of LTI Systems

1) Causality  ⇔  \[ h[n] = 0 \]  for all  \( n < 0 \)

2) Stability  ⇔  \[ \sum_{k=-\infty}^{+\infty} |h[k]| < \infty \]

BIBO — Bounded Input ⇒ Bounded Output

Sufficient condition:  
For \( |x[n]| \leq x_{\text{max}} < \infty \).

\[ |y[n]| = \left| \sum_{k=-\infty}^{\infty} x[k] h[n-k] \right| \leq x_{\text{max}} \sum_{k=-\infty}^{\infty} |h[n-k]| < \infty. \]

Necessary condition:  
If \( \sum_{k=-\infty}^{\infty} |h[k]| = \infty \)

Let \( x[n] = h[-n] / |h[-n]| \), then \( |x[n]| = 1 \) bounded

But  \( y[0] = \sum_{k=-\infty}^{\infty} x[k] h[-k] = \sum_{k=-\infty}^{\infty} h^*[-k] h[-k] / |h[-k]| = \sum_{k=-\infty}^{\infty} |h[-k]| = \infty \)
Next lecture covers:
O & W pp. 90-102, 127-136