Today’s Agenda

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- Complex Number Tricks
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1 The Big Picture Thus Far

Let’s summarize 6.003 so far without using any equations (English only). In Chapter 1 of the textbook, we defined a signal to be a set of complex numbers indexed by time, and the index can be continuous (CT) or discrete (DT). We then defined a system to be a deterministic mapping from input signals to output signals. There are four (independent) system properties of interest to us: linearity, time-invariance, causality and stability.

In Chapter 2, we restricted ourselves to systems that are both linear and time-invariant (LTI). LTI systems are associative, commutative and distributive. Four questions motivated us:

- How can we represent LTI systems?
- Given a representation and an input, how can we find the output?
- How can we convert one representation to another?
- Which representation is the best?

The first characterization of LTI systems was the impulse response, defined to the output of an LTI system when the input is a unit impulse. The impulse response is a complete characterization of an LTI system, namely there is a one-to-one correspondence between the set of LTI systems and the set of impulse responses. The I/O signals and the impulse response are related in a straightforward manner: the output is the convolution sum or convolution integral of the impulse response and the input signal. However, this computation is considered tedious and does not offer much insight into solving certain kinds of problems.

We then looked at another method of describing LTI systems: linear constant coefficient ordinary differential (or difference) equations. Keep in mind that only some LTI systems can be described in this manner, but a large number of the LTI systems that occur in nature fall into this class. The caveat to this description is that diff eq’s do not uniquely specify LTI systems. Two or more LTI systems may be described by exactly the same diff eq. If we specify auxiliary properties, such as stability or causality, then the system is unique. However, finding the output given a particular input is just as tedious as convolution: we need to solve a differential equation. Can we convert from the diff eq (with additional conditions) representation to the impulse response representation? Two problems on Problem Set 2 were devoted to the forward conversion (which was tedious), and we haven’t looked at going in the other direction.

In Chapter 3, we studied what is perhaps the most important idea in all of 6.003: complex exponentials are the eigenfunctions of all LTI systems. In other words, the output of an LTI system when the input is a complex exponential is the same complex exponential scaled by a complex factor. When we restricted the exponents to be purely imaginary, that scale factor, or the eigenvalue corresponding to that eigenfunction, is defined to be the frequency response of the LTI system.\(^1\) Thus, we have a third way of specifying LTI systems: the frequency response. However, this representation can only be used for stable LTI systems (and a few exceptional pathological systems). Now, how do we use the frequency response to find outputs from inputs? Well, if we can express an input as the superposition of complex exponentials, then the output would be the superposition of the same exponentials, each scaled by the frequency response evaluated at that frequency. In Chapter 3, we restricted ourselves to periodic input signals,\(^2\) so we developed the Fourier series representation of signals. In Chapter 4, we generalized to aperiodic signals and used the Fourier transform\(^3\) representation.

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\(^1\) Later, when we relax this restriction, we will call the eigenvalue the transfer function or system function.

\(^2\) Subject to the Dirichlet conditions, that is.

\(^3\) Fourier transforms are not on Quiz 1.
How is the frequency-domain representation related to the others? We can translate diff eq’s into a frequency response by the steps detailed in Tutorial 4. The time and frequency relationship is quite remarkable: the frequency response is the Fourier transform of the impulse response.

Three Ways to Characterize LTI Systems So Far:

- Impulse response: complete characterization, all LTI systems have one.
- Frequency response: only for stable systems (with a few exceptions, like an integrator).
- Diff Eq: only certain LTI systems can be described this way, plus the diff eq alone does not specify the system; need additional info like stability and causality.

Depending on what we are trying to analyze, one of the characterizations above may be better suited. If we are modelling physical phenomena, such as a mechanical system or circuit, it may be easy to derive a differential equation to model that system. From that differential/difference equation we can then solve for the impulse response or the frequency response. We saw that working with the impulse response can be difficult (i.e. convolution). Recently, with the FT we have begun to see that using the frequency domain can turn problems involving convolution into ones involving simple multiplication. In other words, convolution in one domain corresponds to multiplication in the other domain.
<table>
<thead>
<tr>
<th>Representation</th>
<th>Applicable to All LTI Systems?</th>
<th>Is It a Unique and Complete Description?</th>
<th>How to Find Output Given Input</th>
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<tbody>
<tr>
<td>Impulse response</td>
<td>Yes</td>
<td>Yes</td>
<td>Convolution</td>
</tr>
<tr>
<td>Diff eq</td>
<td>No, but many useful ones</td>
<td>No, auxiliary conditions required</td>
<td>Solve the diff eq (tedious)</td>
</tr>
<tr>
<td>Frequency response</td>
<td>No, stable systems only</td>
<td>Yes</td>
<td>Scale each Fourier series coef by frequency response</td>
</tr>
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Table 1: Comparison chart of LTI system representations

2 Complex Number Review

Just to review, here are some basic concepts about complex numbers.

2.1 Rectangular and polar coordinates

Let $z = x + jy = re^{j\theta}$, where $r > 0$. Then the rectangular coordinates can also be written

\[
\begin{align*}
Re\{z\} &= x \quad \text{(real part of } z) \\
Im\{z\} &= y \quad \text{(imaginary part of } z)
\end{align*}
\]

and the polar coordinates can be written

\[
\begin{align*}
r &= |z| \quad \text{(magnitude of } z) \\
\theta &= \angle z \quad \text{(phase of } z)
\end{align*}
\]

Converting between rectangular and polar forms:

\[
\begin{align*}
r &= \sqrt{x^2 + y^2} \\
\theta &= \tan^{-1}(y/x) \\
x &= r \cos(\theta) \\
y &= r \sin(\theta)
\end{align*}
\]

Think of $z$ as the vector $(x, y)$ in $\mathbb{R}^2$ to see these. Here $\tan^{-1}(y/x)$ is the multivalued inverse of the tangent function, and the particular value that we choose for the phase of $z$ will be sensitive to the quadrant in which $z$ lies. For example, $\angle(-1 + j) = \tan^{-1}(-1/1) = -\pi/4$ but $\angle(1 - j) = \tan^{-1}(1/-1) = 3\pi/4$. In other words, $\angle z$ is the angle made by $(x, y)$ with the $x$-axis, or equivalently, the angle $\theta$ such that $\cos(\theta) = x/r, \sin(\theta) = y/r$.

2.2 Complex conjugates

The complex conjugate of $z$, denoted as $z^*$, is the complex number that results from reflecting $z$ across the real axis. Thus, the real parts and magnitudes of $z$ and $z^*$ are the same; the imaginary parts and angles of $z$ and $z^*$ are opposites:

\[
\begin{align*}
Re\{z^*\} &= Re\{z\} \\
Im\{z^*\} &= -Im\{z\} \\
|z^*| &= |z| \\
\angle(z^*) &= -\angle z
\end{align*}
\]
i.e., if \( z = x + jy = re^{j\theta} \) then \( z^* = x - jy = re^{-j\theta} \). If \( z \) is expressed in a form other than explicitly rectangular or polar, we can get \( z^* \) by negating every instance of \( j \) in the expression for \( z \). For example, if \( z = \frac{(1+e^{j\omega})}{(1-j\omega)} \), then \( z^* = \frac{1+j\omega}{1+e^{j\omega}} \). From a complex number and its conjugate, we can extract the real and imaginary parts, magnitude, and phase:

\[
\begin{align*}
Re\{z\} & = \frac{1}{2}(z + z^*) \\
Im\{z^*\} & = \frac{1}{2j}(z - z^*) \\
|z| & = (zz^*)^{\frac{1}{2}} \\
\angle z & = \frac{\ln(z/z^*)}{2j}
\end{align*}
\]

One interesting property about the conjugation operator is that it distributes over nearly everything:

\[
\begin{align*}
(x + y)^* & = x^* + y^* \\
(xy)^* & = x^*y^* \\
(x^y)^* & = (x^*)^y, \text{ if we allow multiple interpretation of roots when } y \text{ is a fraction.} \\
(\log_x y)^* & = \log_{x^*} y^*, \text{ same issue as the previous one.} \\
(cos x)^* & = \cos(x^*)
\end{align*}
\]

To convince ourselves that the last formula \((\cos x)^* = \cos(x^*)\) (as well as similar ones) is true, we can expand the function as a Taylor series, and distribute the conjugation over more familiar operations:

\[
\begin{align*}
(\cos x)^* & = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right)^* \\
& = 1 - \frac{(x^*)^2}{2!} + \frac{(x^*)^4}{4!} - \cdots \\
& = \cos(x^*)
\end{align*}
\]

There is no fundamental difference between \( j \) and \(-j\), and that explains why the conjugate (almost) always distribute. We just arbitrarily chose \( j = \sqrt{-1} \) from one of two possible values. However, the very act of defining \( j = \sqrt{-1} \) is what creates some exceptions. For example:

\[
(\sqrt{-1})^* = -j \neq j = \sqrt{-1^*}
\]

This is an exception to the \((xy)^* = (x^*)^y^*\) rule, where \( x = -1 \) and \( y = 1/2 \). The logarithm rule also has a similar issue; for example, check out \( \log_{-1}(j) = 1/2 \).

If we say that \( \sqrt{-1} \) could be \( j \) or \(-j\), then the conjugate should always distribute.

### 2.3 Euler’s and related relations

Here is Euler’s relation:

\[
e^{j\theta} = \cos \theta + j \sin \theta
\]
which implies
\[
\begin{align*}
\cos \theta &= \frac{e^{j\theta} + e^{-j\theta}}{2} \\
\sin \theta &= \frac{e^{j\theta} - e^{-j\theta}}{2j}
\end{align*}
\]

The Euler relation can be derived by showing that the Taylor series expansion of both sides are equal.

### 3 Complex Number Tricks

#### 3.1 Evaluating the magnitude and phase of a sum of complex exponentials

Suppose we wanted to find the magnitude and phase of the signal:
\[
x[n] = e^{j(-\frac{\pi}{6}n - \frac{\pi}{2})} + \frac{1}{2} e^{j\frac{\pi}{3}n} + 3e^{j\frac{\pi}{6}n} + \frac{1}{2} e^{j\frac{5\pi}{6}n} + e^{j(\frac{\pi}{3}n + \frac{\pi}{2})}.
\]

This form of a signal comes up all the time, and we would like a quick and neat method to do this. Let’s rearrange and group the terms:
\[
x[n] = 3e^{j\frac{\pi}{2}n} + \frac{1}{2} \left(e^{j\frac{\pi}{3}n} + e^{j\frac{5\pi}{6}n}\right) + \left(e^{j(\frac{\pi}{3}n + \frac{\pi}{2})} + e^{j(-\frac{\pi}{3}n - \frac{\pi}{2})}\right).
\]

Note that in each group, the average of the exponents is \(e^{j\frac{\pi}{2}n}\), so let’s factor it out:
\[
x[n] = e^{j\frac{\pi}{2}n} \left[3 + \frac{1}{2} (e^{-j\frac{\pi}{3}n} + e^{j\frac{\pi}{3}n}) + \left(e^{j(\frac{\pi}{3}n + \frac{\pi}{2})} + e^{j(-\frac{\pi}{3}n - \frac{\pi}{2})}\right)\right].
\]

We can factor out the \(\frac{\pi}{2}\) phase, which becomes \(j\):
\[
x[n] = e^{j\frac{\pi}{2}n} \left[3 + \frac{1}{2} (e^{-j\frac{\pi}{3}n} + e^{j\frac{\pi}{3}n}) + \left(e^{j(\frac{2\pi}{3}n)} + e^{j(-\frac{2\pi}{3}n)}\right)\right]
\]

We apply the Euler relations for cosine and sine:
\[
x[n] = e^{j\frac{\pi}{2}n} \left[3 + \frac{1}{2} \cos \left(\frac{\pi}{3} n\right) - 2 \sin \left(\frac{2\pi}{3} n\right)\right].
\]

\(x[n]\) is now in the form \(x[n] = r[n]e^{j\theta[n]}\), where \(r[n]\) and \(\theta[n]\) are the magnitude and phase, respectively:
\[ r[n] = 3 + \frac{1}{2} \cos \left( \frac{\pi}{3} n \right) - 2 \sin \left( \frac{2\pi}{3} n \right), \]
\[ \theta[n] = \frac{\pi}{2} n. \]

We always have to double-check that \( r[n] \) is in fact always nonnegative, since a magnitude cannot be negative. If \( r[n] \) takes on negative values, then we need to add \( \pi \) to \( \theta[n] \).

### 3.2 Evaluating the magnitude and phase of a frequency response

As we saw in Tutorial 4, we sometimes want to find the magnitude and phase of CT frequency responses. In Problem 4.1, we have the following frequency response:

\[ H(j\omega) = \frac{1}{-\omega^2 - \frac{3}{2} + \frac{5}{2} j \omega}. \]

In part (b-i), we need to evaluate the magnitude and phase when \( \omega = 3\pi \):

\[ H(j3\pi) = \frac{1}{-(3\pi)^2 - \frac{3}{2} + \frac{5}{2} j (3\pi)} = \frac{2}{-18\pi^2 - 3 + j15\pi}. \]

In general we can evaluate the magnitude and phase of a complex number, \( c \), as follows

\[ |c| = \sqrt{c^*c} = \sqrt{\text{Re}\{c\}^2 + \text{Im}\{c\}^2} \]
\[ \angle c = \tan^{-1}\left( \frac{\text{Im}\{c\}}{\text{Re}\{c\}} \right). \]

Oftentimes, we need to find the magnitude and phase of a ratio of complex numbers, \( a \) and \( b \),

\[ \left| \frac{a}{b} \right| = \frac{|a|}{|b|} \]
\[ \angle \left( \frac{a}{b} \right) = \angle a - \angle b \]

Applying these results to the example above, we have

\[ |H(j3\pi)| = \frac{2}{\sqrt{(18\pi^2 + 3)^2 + (15\pi)^2}} \]
\[ \angle H(j3\pi) = -\tan^{-1}\left( \frac{15\pi}{-18\pi^2 - 3} \right) = \pi + \tan^{-1}\left( \frac{5\pi}{6\pi^2 + 1} \right) \]
For DT LTI systems described by difference equations, we should use the conjugate to find the magnitude of the frequency response. For example, the stable DT LTI system described by the difference equation:

\[ y[n] + 2y[n - 1] = 3x[n] \]

has the frequency response:

\[ H(e^{j\omega}) = \frac{3}{1 + 2e^{-j\omega}}. \]

The magnitude is:

\[
|H(e^{j\omega})| = \sqrt{H^*(e^{j\omega})H(e^{j\omega})} = \sqrt{\frac{3}{1 + 2e^{j\omega}} \cdot \frac{3}{1 + 2e^{-j\omega}}} = \frac{3}{\sqrt{1 + 2e^{-j\omega} + 2e^{j\omega} + 4}} = \frac{3}{\sqrt{5 + 4\cos(\omega)}}
\]

### 3.3 Real/imaginary parts and even/odd parts

A function \( f(x) \) is conjugate even if we get the same thing after flipping it across the vertical axis and conjugating it:

\[ f(x) = f^*(-x) \quad \text{(Conjugate even function)} \]

A function \( f(x) \) is conjugate odd if we get the same thing after flipping it across the vertical axis, negating it, and conjugating it:

\[ f(x) = -f^*(-x) \quad \text{(Conjugate odd function)} \]

“Regular” (non-conjugate) evenness and oddness are similar, just with the conjugation operator removed. These concepts appear in 6.003 when we talk about Fourier series and Fourier transforms: If a signal is real in one domain, it is conjugate even in the other domain. If a signal is imaginary in one domain, it is conjugate odd in the other domain. In particular, we deal with real signals all the time, which means that the Fourier series coefficients are conjugate even:

\[ a_k = a_{-k}^* \quad \text{(For real signals)} \]

We can see that this means that \( \text{Re}\{a_k\} \) and \( |a_k| \) are even, \( \text{Im}\{a_k\} \) and \( \angle a_k \) are odd.

As another example, let’s consider a periodic signal \( x(t) \) that is real and odd. From the rule above, this means that its Fourier series coefficients \( a_k \) are imaginary and conjugate even (for real signals, conjugate oddness is equivalent to oddness). Can we simplify this representation? We see above that the imaginary part of \( a_k \) is odd. Since \( a_k \) is purely imaginary anyway, this means that if something is purely imaginary and conjugate even, then it is odd (in the the non-conjugate sense).
4 Sums and Integrals

The following are some common sums and integrals that you may find useful.

- The finite sum formula:
  \[ \sum_{n=0}^{N-1} \alpha^n = \begin{cases} N, & \alpha = 1 \\ \frac{N}{1-\alpha}, & \text{for any complex number } \alpha \neq 1 \end{cases} \]

- The infinite sum formula: if \(|\alpha| < 1\), then
  \[ \sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha} \]

- Orthogonality of complex exponentials (we will explain these in more detail in Tutorial 6):
  \[
  \delta(t - t') = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{j\omega(t-t')} \, d\omega \\
  \delta[n - n'] = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j\omega(n-n')} \, d\omega \\
  \delta(\omega - \omega') = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{j(\omega-\omega')t} \, dt \\
  \delta[k - k'] = \frac{1}{T} \int_{-\pi}^{+\pi} e^{j(k-k')(2\pi/T)t} \, dt \\
  \sum_{m=-\infty}^{+\infty} \delta(t - t' - mT) = \frac{1}{T} \sum_{k=-\infty}^{+\infty} e^{j(k-t')2\pi/T} \\
  \sum_{m=-\infty}^{+\infty} \delta[n - n' - mN] = \frac{1}{N} \sum_{k=(N)}^{+\infty} e^{j(k(n-n'))2\pi/N} \\
  \sum_{m=-\infty}^{+\infty} \delta(\omega - \omega' - m2\pi) = \frac{1}{2\pi} \sum_{n=-\infty}^{+\infty} e^{j(\omega-\omega')n} \\
  \sum_{m=-\infty}^{+\infty} \delta[k - k' - mN] = \frac{1}{N} \sum_{n=(N)}^{+\infty} e^{j(n(k-k'))2\pi/N} \\
  \]

- The finite integral of the square of sines and cosines (“Fourier’s trick”):
  \[ \int_0^1 \sin^2(\pi x) \, dx = \int_0^1 \cos^2(\pi x) \, dx = \frac{1}{2} \]

5 Trigonometric Identities

- Angle addition and subtraction formulas
\[
\begin{align*}
\sin(x \pm y) &= \sin(x) \cos(y) \pm \cos(x) \sin(y) \\
\cos(x \pm y) &= \cos(x) \cos(y) \mp \sin(x) \sin(y)
\end{align*}
\]

- **Power reduction formulas**

\[
\begin{align*}
\cos^2(x) &= \frac{1 + \cos(2x)}{2} \\
\sin^2(x) &= \frac{1 - \cos(2x)}{2} \\
\cos^3(x) &= \frac{3 \cos(x) + \cos(3x)}{4} \\
\sin^3(x) &= \frac{3 \sin(x) - \sin(3x)}{4}
\end{align*}
\]

- **Half-angle formulas**

\[
\begin{align*}
\cos\left(\frac{x}{2}\right) &= \pm \sqrt{\frac{1 + \cos(x)}{2}} \\
\sin\left(\frac{x}{2}\right) &= \pm \sqrt{\frac{1 - \cos(x)}{2}}
\end{align*}
\]

- **Product-to-sum identities**

\[
\begin{align*}
\cos(x) \cos(y) &= \frac{\cos(x + y) + \cos(x - y)}{2} \\
\sin(x) \sin(y) &= \frac{\cos(x - y) - \cos(x + y)}{2} \\
\sin(x) \cos(y) &= \frac{\sin(x + y) + \sin(x - y)}{2} \\
\cos(x) \sin(y) &= \frac{\sin(x + y) - \sin(x - y)}{2}
\end{align*}
\]

- **Sum-to-product identities**

\[
\begin{align*}
\cos(x) + \cos(y) &= 2 \cos\left(\frac{x + y}{2}\right) \cos\left(\frac{x - y}{2}\right) \\
\sin(x) + \sin(y) &= 2 \sin\left(\frac{x + y}{2}\right) \cos\left(\frac{x - y}{2}\right) \\
\cos(x) - \cos(y) &= -2 \sin\left(\frac{x + y}{2}\right) \sin\left(\frac{x - y}{2}\right) \\
\sin(x) - \sin(y) &= 2 \cos\left(\frac{x + y}{2}\right) \sin\left(\frac{x - y}{2}\right)
\end{align*}
\]
6 Caveats About the Unit Impulse

6.1 Time scaling

What happens if we time scale a unit impulse? Watch out! The result is different for DT and CT. For an integer \( k \), there is no change in DT:

\[
\delta[kn] = \delta[n].
\]

However, in CT and real scale factor \( a \):

\[
\delta(at) = \frac{1}{|a|} \delta(t).
\]

In DT, there is no additional scale factor in front of the delta function, but it appears in CT because this factor refers to the area of the impulse when integrated.

6.2 Differentiation and the product rule

We know that if the CT unit impulse is multiplied by a signal, we can replace the signal with its value at the time that the delta function is nonzero.

For example:

\[
x(t)\delta(t - t_0) = x(t_0)\delta(t - t_0)
\]

Similarly we can try to compute derivatives of expressions involving delta functions by using the product rule. However, what is interesting that the following work

\[
\frac{d}{dt} f(t)\delta(t) = f'(t)\delta(t) + f(t)u_1(t) = f'(0)\delta(t) + f(0)u_1(t)
\]

is incorrect, but

\[
\frac{d}{dt} f(t)\delta(t) = \frac{d}{dt} f(0)\delta(t) = f(0)u_1(t)
\]

is correct. You can think of the doublet as “over-powering” the delta function. We can avoid these complications by simplifying all expressions at each stage of manipulation involving delta functions.

7 Caveats About the CT and DT Fourier Series

7.1 Specifying the period

In 6.003, we define the Fourier series to be in terms of the fundamental period of the signal. Thus, given a set of Fourier series coefficients in DT, we can find the signal uniquely (both the signal \( x[n] \) and the FS coefficients \( a_k \) have the same period \( N \)). However, given only a set of Fourier series coefficients in CT and nothing else, we cannot find the corresponding signal without knowing the fundamental period.

When asked to “find the Fourier series coefficients” (on an exam, for instance), we should always use the fundamental period so that the answer is unique. That being said, it is sometimes useful to use a Fourier series expansion with respect to a period of the signal that is not the fundamental period, especially when combining signals that have different fundamental periods. We will see examples of this later on in this section.
7.2 Differentiation rule in CT

Note that the differentiation property for CT FS states that:

\[ x(t) \longleftrightarrow a_k \]
\[ y(t) = \frac{d}{dt} x(t) \longleftrightarrow b_k = j \kappa \omega_0 a_k \]

Therefore, \( b_0 = 0 \).

The derivative of a periodic CT signal has no DC value, i.e. its Fourier series coefficient \( a_0 \) is zero.

When going in the reverse direction (i.e. integrating) we must take care to calculate the \( a_0 \) term separately.

7.3 Time scaling

Time scaling a CT signal by \( a \) doesn’t change its FS representation. However, in DT, time expansion by an integer \( m \) gives a height scaling of \( \frac{1}{m} \). Consider \( a_0 \) before the time expansion

\[ a_0 = \frac{1}{N} \sum_{n=(N)} x[n] \]

Now, the time scaled version, \( x'[n] \), where

\[ x'[n] = \begin{cases} x[n/m] & \text{if } n \text{ is a multiple of } m \\ 0 & \text{if } n \text{ is not a multiple of } m \end{cases} \]

has period \( mN \) and

\[ a'_0 = \frac{1}{mN} \sum_{n=(mN)} x'[n] = \frac{1}{mN} \sum_{n=(N)} x[n] = \frac{1}{m} a_0 \]

One way of thinking about this is that in CT, we “fill in the gaps” perfectly, while in DT, we leave “holes” so there is somehow less signal than before.

7.4 Frequency scaling

Let \( x[n] \) be a periodic DT signal with Fourier series coefficients \( a_k \). Let \( y[n] \) be the signal whose Fourier series coefficients \( b_k \) is related to those of \( a_k \) for some integer \( m \), as:

\[ b_k = \begin{cases} a_{k/m}, & \text{if } k \text{ is a multiple of } m \\ 0, & \text{if } k \text{ is not a multiple of } m \end{cases} \]

What is relationship between \( x[n] \) and \( y[n] \)?
First we note that $b_k$ consists of the coefficients $a_k$ spread out by a factor of $m$, this means that we must sum over a period of $N' = mN$ to include all the coefficients. Thus,

$$y[n] = \sum_{k=(mN)} b_k e^{-j k \frac{2\pi}{mN} n}$$

setting $k = lm$ gives

$$= \sum_{l=(N)} b_{lm} e^{-j lm \frac{2\pi}{mN} n}$$

$$= \sum_{l=(N)} a_l e^{-j l \frac{2\pi}{N} n}$$

$$= x[n]$$

In other words, we have the same signal. The reason we have two different sets of Fourier series coefficients is that for the second set, we are using $mN$, a multiple of the fundamental period $N$, as the period. Using this interpretation, we can derive the original relationship between $a_k$ and $b_k$ from the analysis equation:

$$b_k = \frac{1}{mN} \sum_{n=(mN)} y[n] e^{-j k (2\pi/mN) n}$$

$$= \frac{1}{mN} \sum_{n=(mN)} x[n] e^{-j k (2\pi/mN) n}$$

$$= \frac{1}{mN} \sum_{n=(mN)} \left( \sum_{l=(N)} a_l e^{j (2\pi/N) n} \right) e^{-j k (2\pi/mN) n}$$

$$= \sum_{l=(N)} a_l \left( \frac{1}{mN} \sum_{n=(mN)} e^{j (ml-k)(2\pi/mN) n} \right)$$

$$= \sum_{l=(N)} a_l \delta[ml - k]$$

$$= \begin{cases} 
 a_k/m, & \text{if } k \text{ is a multiple of } m \\
 0, & \text{if } k \text{ is not a multiple of } m
\end{cases}$$

The same is also true in CT.

### 7.5 Multiplication by $(-1)^n$ and $(-1)^k$

Let $x[n]$ be a periodic DT signal with period $N$ and Fourier series coefficients $a_k$. Let $y[n] = (-1)^n x[n]$, and let $b_k$ be the Fourier series coefficients of $y[n]$. What’s the relationship between $b_k$ and $a_k$? When we multiply two signals in the time domain, we can only use the properties if they have the same period. If they do not, then we’ll have to use the lowest common multiple as the new period. In this case, the period of $y[n]$ is $\text{LCM}(N, 2)$, which is $N$ when $N$ is even and $2N$ when $N$ is odd.

We have to consider two cases: when $N$ is even and when it is odd. Let’s look at the even case first. We can rewrite $(-1)^n$:

$$(-1)^n = e^{j \pi n} = e^{j \pi/2 (N/2) n}.$$
From the Table of DTFS Properties, we see that this corresponds to a shift in the $k$-axis by $N/2$:

$$b_k = a_{k \pm \frac{N}{2}}.$$  

Thus, we are shifting $x[n]$ by half a period. Since $x[n]$ is periodic, it does not matter which direction we shift it. Now, let’s consider the odd case. We clearly cannot use the expression above because $N/2$ is not an integer. Keep in mind that we can use any period of $x[n]$ to form a Fourier series representation. Let’s use $2N$ as the period; call the corresponding coefficients $a'_k$. As we saw from the previous subsection, this means that:

$$a'_k = \begin{cases} a_k/2, & \text{if } k \text{ is a multiple of 2} \\ 0, & \text{if } k \text{ is not a multiple of 2} \end{cases}$$

We instead rewrite $(-1)^n$ using $2N$ as the period:

$$(-1)^n = e^{\pm j\pi n} = e^{\pm j(N/2N)n}.$$ 

Now, we are shifting $a'_k$ by $N$ to form $b_k$:

$$b_k = a'_{k-N} = \begin{cases} a_{(k-N)/2}, & \text{if } (k-N) \text{ is a multiple of 2} \\ 0, & \text{if } (k-N) \text{ is not a multiple of 2} \end{cases}$$

Now, what if we consider the opposite situation? Namely, Let $x[n]$ be a periodic DT signal with period $N$ and Fourier series coefficients $a_k$. Let $c_k = (-1)^ka_k$, and let $z[n]$ be the signal that results from the coefficients $c_k$. What’s the relationship between $z[n]$ and $x[n]$? As we would expect, the answer is similar to that above, though with one difference in the odd case. You can check that when $N$ is even, we have:

$$z[n] = x\left[n \pm \frac{N}{2}\right].$$

However, when $N$ is odd, we get:

$$z[n] = \begin{cases} 2x\left[\frac{n-N}{2}\right], & \text{if } (n-N) \text{ is a multiple of 2} \\ 0, & \text{if } (n-N) \text{ is not a multiple of 2} \end{cases}$$

There is extra factor of 2 that pops up from the math. One way to see this is from the earlier subsection on time scaling in DT. Another way to see why this comes in is to look at Parseval’s relation:

$$\frac{1}{N} \sum_{n=-N}^{N} |x[n]|^2 = \sum_{k=-N}^{N} |a_k|^2$$

In the first case, multiplying the time signal by $(-1)^n$ and doubling the period does not increase the left-hand side of the equation because we sum twice as much signal but we also divide by a period twice as large (and the absolute value of $(-1)^n$ is 1). This means that the sum on the right-hand side is also preserved, which is what we get by “expanding” $a_k$ on the $k$-axis and shifting it. In the second case, we are multiplying the $a_k$’s by $(-1)^k$ and doubling the period, which does double the right-hand side. Thus, the left-hand side is also doubled. Since we divide the left-hand sum by a number twice as big, the sum of the square of the absolute value of the signal must quadruple. Of course, there are many ways this could be achieved, and scaling $x[n]$ by 2 is one such method. We are not proving from this explanation that this is the only solution (which we can achieve from the synthesis and analysis equations), but rather we are just showing that it is consistent with Parseval’s relation.
This interesting exercise has no counterpart in CT.

7.6 Matrix view of the DTFS (optional)

We can view the operations of the analysis and synthesis equations for DTFS in terms of matrix operations and transformations in vector space. Consider the $N \times N$ matrix $A$ defined as:

\[
A \triangleq \begin{pmatrix}
e^{-j\omega_0 \cdot 0} & e^{-j\omega_0 \cdot 0 \cdot 1} & \cdots & e^{-j\omega_0 \cdot 0 \cdot (N-1)} \\
e^{-j\omega_0 \cdot 1 \cdot 0} & e^{-j\omega_0 \cdot 1 \cdot 1} & \cdots & e^{-j\omega_0 \cdot 1 \cdot (N-1)} \\
\vdots & \vdots & \ddots & \vdots \\
e^{-j\omega_0 \cdot (N-1) \cdot 0} & e^{-j\omega_0 \cdot (N-1) \cdot 1} & \cdots & e^{-j\omega_0 \cdot (N-1) \cdot (N-1)} \\
\end{pmatrix}
\]

Now consider two $N \times 1$ vectors $x$ and $a$ representing one period of a DT signal and its corresponding FS coefficients. Then, we can write the synthesis and analysis equations as a linear transformation from one vector to the other:

\[
a = \frac{1}{N}Ax \\
x = A^\dagger a
\]

where $(\cdot)^\dagger$ represents conjugate transpose.

Note that $A^\dagger A = AA^\dagger = NI$, where $I$ is the identity matrix. Thus, $\frac{1}{\sqrt{N}}A$ is a unitary matrix and represents an orthonormal transformation. If we remove the $\frac{1}{\sqrt{N}}$ scale factor, then $A$ is an orthogonal (but not orthonormal) transformation. This reinforces the idea the the Fourier series is nothing more than an orthogonal change of coordinate system.

It is now easy to prove Parseval’s relation. The square of the length a vector is the inner product of the vector with itself, or the sum of the squares of each orthogonal component:

\[
\sum_{n=\langle N \rangle} |x[n]|^2 = x^\dagger x = (A^\dagger a)^\dagger (A^\dagger a) = (a^\dagger A) (A^\dagger a) \\
= a^\dagger (AA^\dagger) a = a^\dagger (NI) a = Na^\dagger a \\
= N \sum_{k=\langle N \rangle} |a_k|^2
\]

Thus:

\[
\frac{1}{N} \sum_{n=\langle N \rangle} |x[n]|^2 = \sum_{k=\langle N \rangle} |a_k|^2.
\]

8 Eigenstuff

We showed in lecture that a certain set of input signals, namely complex exponentials of the form $x(t) = e^{st}$ ($x[n] = z^n$), are eigenfunctions of LTI systems, i.e. the corresponding outputs are simply scaled versions of inputs of this form, and this scaling factor is the eigenvalue. We showed that the outputs of CT and DT
LTI systems in response to \( x(t) = e^{st} \) and \( x[n] = z^n \) are \( y(t) = H(s)e^{st} = H(s)x(t) \) and \( y[n] = H(z)z^n = H(z)x[n] \), respectively, where the eigenvalues \( H(s) \) and \( H(z) \) associated with the given eigenfunctions are:

\[
H(s) = \int_{-\infty}^{+\infty} h(\tau)e^{-s\tau} \, d\tau \\
H(z) = \sum_{k=-\infty}^{+\infty} h[k]z^{-k}
\]

where \( h(t) \) and \( h[n] \) are the impulse responses of the systems.

The superposition property of LTI systems suggests another way of writing signals. We can express an input signal as a linear combination of complex exponentials. Then, the output of an LTI system is the same linear combination of the exponentials scaled by the appropriate eigenvalue. So, if the inputs are:

\[
x(t) = \sum_{k} a_k e^{s_k t} \\
x[n] = \sum_{k} a_k z_k^n
\]

then the outputs are:

\[
y(t) = \sum_{k} a_k H(s_k) e^{s_k t} \\
y[n] = \sum_{k} a_k H(z_k) z_k^n
\]
Problem 5.1

Consider a linear system $H$ that has input-output pairs depicted in the figure below. Determine the following and explain your answers:

(a) Is this system causal?

(b) Is this system time invariant?
Problem 5.2

The systems given below have input $x(t)$ or $x[n]$ and output $y(t)$ or $y[n]$, respectively. Determine whether each of them is (i) stable, (ii) causal, (iii) linear, and (iv) time invariant.

(a) $y(t) = \int_{-\infty}^{t/2} x(\tau) d\tau$

(b) $y[n] = x[n] \sum_{k=-\infty}^{\infty} \delta[n - 2k]$

(c) $y[n] = \log_{10}(|x[n]|)$
Problem 5.3

Evaluate the following continuous-time convolution integrals given below.

(a) \( y_a(t) = [\cos(\pi t)(u(t + 1) - u(t - 3))] \ast [e^{-2t}u(t)]. \)

(b) \( y_b(t) = [(t + 2t^2)(u(t + 1) - u(t - 1))] \ast 2u(t + 2). \)
Problem 5.4

The convolution of a signal with itself turned around in time is called the *autocorrelation function* of that signal. In continuous-time, the autocorrelation function is given by

\[ r_x(t) = \int_{-\infty}^{\infty} x(\tau)x(\tau - t)d\tau \]

(a) By comparison with the convolution integral, determine the impulse response of a system which, given a particular \( x(t) \) as its input, will yield \( r_x(t) \) as its output. Such a system is called a *matched filter*.

(b) Suppose \( x(t) \) is given by

\[ x(t) = u(t) - 2u(t - 3) + 2u(t - 5) - 2u(t - 6) + u(t - 7), \]

![Graph of x(t)](image)

sketch the impulse response of the associated matched filter.

(c) Sketch the output of the matched filter with \( x(t) \) above as input. That is, sketch the autocorrelation function \( r_x(t) \) corresponding to \( x(t) \).
Problem 5.5

Suppose we have a causal, stable DT LTI system whose input $x[n]$ and output $y[n]$ are related by the difference equation:

$$y[n] - \frac{1}{2}y[n - 1] = x[n].$$

Find the output $y[n]$ of the system when the input $x[n]$ is:

$$x[n] = \sin \left( \frac{2\pi}{3} n \right) + u[n - 2].$$

The four key points to solving this problem are:

- Finding the impulse response $h[n]$ of a LTI system described by a difference equation given the proper conditions,
- Finding the frequency response $H(e^{j\omega})$ of the system from the difference equation,
- Using the eigenfunction property of LTI systems, and
- Using the commutative property of cascaded LTI systems.

(Work space)
(Work space)
Problem 5.6

(a) Find the impulse response $h_a[n]$ of a DT LTI system whose output is $y_a[n] = \delta[n]$ when the input is $x_a[n] = (\frac{1}{3})^n u[n]$.

(b) Find the impulse response $h_b(t)$ of a CT LTI system whose output is $y_b(t) = t[u(t) - u(t - 3)]$ when the input is $x_b(t) = u(t - 1)$.

(c) Find the impulse response $h_c(t)$ of a CT LTI system whose output is $y_c(t) = t[u(t) - u(t - 3)] + 3u(t - 3)$ when the input is $x_c(t) = u(t - 1)$. 
Problem 5.7

Evaluate the following sum:

\[ S = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots \]

You may find the following observation useful. We can also express \( S \) as:

\[ S = -\frac{\pi^2}{8} + \frac{1}{2} \sum_{k=-\infty}^{+\infty} \left( \frac{\sin(k\pi/2)}{k\pi} \right)^2. \]

Then, use Fourier series properties on the sum over \( k \). You can solve it two different ways using either Parseval’s relation or the periodic convolution property. You should find that:

\[ S = \frac{\pi^2}{8}. \]
Problem 5.8

Suppose $x[n]$ is a periodic discrete-time signal with Fourier series coefficients $a_k$. The signal has the following properties:

1. $x[n]$ is real.
2. $x[n]$ is even.
3. The fundamental period of $x[n]$ is $N = 5$.
4. $\sum_{n=-2}^{n=2} x[n] = 0$.
5. $a_6 = 1$.
6. $\frac{1}{2} \sum_{n=0}^{n=4} |x[n]|^2 = 2$.

Find an expression that describes $x[n]$ completely.
(Work space)