Directions: The exam consists of 5 problems on pages 2 to 19. Please make sure you have all the pages. Enter all your work and your answers directly in the spaces provided on the printed pages of this booklet. Please make sure your name is on all sheets. DO IT NOW! All sketches must be adequately labeled. Unless indicated otherwise, answers must be derived or explained, not just simply written down. This examination is closed book, but students may use one 8 1/2 × 11 sheet of paper for reference. Calculators may not be used. Note that the problems are not in the order of difficulty. Solve the problems that you can first.

NAME: ______________________________

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Problem 1  (12 Points)

Consider the following three systems:

- **System A**:  
  \[ y(t) = \text{Re}\{x(t)\} \]

- **System B**:  
  \[ y(t) = x(t) + \frac{dx(t + 1)}{dt} \]

- **System C**:  
  \[ y[n] = e^{j\omega_0(n+1)} x[n] \quad \omega_0 \neq 0 \]

where \( x \) and \( y \) are the input and output of each system, respectively.

Circle YES or NO for each of the following questions for each of these systems (no explanation is required).

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<th>System A</th>
<th>System B</th>
<th>System C</th>
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<tbody>
<tr>
<td>Is the system <strong>linear</strong>?</td>
<td>NO</td>
<td>YES</td>
<td>YES</td>
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<tr>
<td>Is the system <strong>time invariant</strong>?</td>
<td>YES</td>
<td>YES</td>
<td>NO</td>
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<td>Is the system <strong>causal</strong>?</td>
<td>YES</td>
<td>NO</td>
<td>YES</td>
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<td>Is the system <strong>stable</strong>?</td>
<td>YES</td>
<td>NO</td>
<td>YES</td>
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**Note:** System A is non-linear because \( ay(t) \) may not be equal to \( \text{Re}\{ax(t)\} \) when \( a \) is not real.
Problem 2  (20 Points)

The impulse response of an LTI system is given by

\[ h(t) \]

(a) Compute and sketch below the step response \( s(t) \) of the system (i.e. the output when the input is the unit step \( u(t) \)).

The step response is given by

\[ s(t) = h(t) * u(t) = \int_{-\infty}^{t} h(\tau)d\tau \]

Therefore,
(b) One cycle of a sine wave $x(t)$ (shown below) is the input to the system. Compute and sketch the corresponding output $y(t)$.

\[
x(t) = \begin{cases} \sin(\pi t) & 0 \leq t < 2 \\ 0 & \text{o.w.} \end{cases}
\]

\[
h(t) = \begin{cases} 2 & 1 \leq t < 3 \\ 0 & \text{o.w.} \end{cases}
\]

We can take the derivative of $h(t)$ and convolve $h'(t)$ with $x(t)$ to get $z(t)$

\[
h'(t) = 2 \ast \delta(t - 1) - 2 \ast \delta(t - 3)
\]

\[
z(t) = x(t) \ast h'(t) = 2x(t - 1) - 2x(t - 3)
\]

Now,

\[
y(t) = \int_{-\infty}^{t} z(\tau)d\tau = \begin{cases} \begin{cases} 0 & t < 1 \\ 2 \int_{1}^{t} \sin(\pi(\tau - 1))d\tau & 1 \leq t < 3 \\ -2 \int_{3}^{t} \sin(\pi(\tau - 3))d\tau & 3 \leq t < 5 \\ 0 & t \geq 5 \end{cases} & \end{cases}
\]

You could also do it directly by using the convolution formula

\[
y(t) = h(t) \ast x(t) = \int_{-\infty}^{\infty} h(\tau)x(t - \tau)d\tau = \int_{1}^{3} 2x(t - \tau)d\tau = \begin{cases} \begin{cases} 0 & t < 1 \\ 2 \int_{1}^{3} \sin(\pi(\tau - 1))d\tau & 1 \leq t < 3 \\ 2 \int_{3}^{t} \sin(\pi(\tau - 3))d\tau & 3 \leq t < 5 \\ 0 & t \geq 5 \end{cases} & \end{cases}
\]

\[
= \begin{cases} \begin{cases} 0 & t < 1 \\ 2 \frac{(1 - \cos(\pi(t - 1)))}{\pi} & 1 \leq t < 3 \\ 2 \frac{(1 - \cos(\pi(t - 3)))}{\pi} & 3 \leq t < 5 \\ 0 & t \geq 5 \end{cases} & \end{cases}
\]
Problem 3 (21 Points)

Consider the signal \( x[n] \) shown below:

(a) What is the maximum number of independent Fourier series coefficients that this signal can have?

Maximum number = 3

Brief explanation:
The fundamental period is 5. Since \( x[n] \) is even and real, \( a_k \) should be even and real. Thus, the maximal number should be 3. It can not be greater than 3. But it can be less.
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(b) Derive the Fourier series coefficients $a_k$ of $x[n]$.

$$a_k = \frac{1}{5} \sum_{-2}^{2} x[n] e^{-j \frac{2\pi}{5} n} = \begin{cases} \frac{1}{5} + \frac{3}{5} \cos \left( \frac{2\pi}{5} \right) - \frac{4}{5} \cos \left( \frac{4\pi}{5} \right) & k = 5m \quad \forall m \in Z \\ \frac{1}{5} + \frac{3}{5} \cos \left( \frac{4\pi}{5} \right) - \frac{4}{5} \cos \left( \frac{2\pi}{5} \right) & k = 5m \pm 1 \quad \forall m \in Z \\ \frac{1}{5} + \frac{3}{5} \cos \left( \frac{2\pi}{5} \right) - \frac{4}{5} \cos \left( \frac{4\pi}{5} \right) & k = 5m \pm 2 \quad \forall m \in Z \end{cases}$$

There exist multi-expressions of $a_k$. For example:

$$a_k = \frac{1}{5} \sum_{-2}^{2} (-2) e^{-j \frac{2\pi}{5} n} + \frac{1}{5} \sum_{-1}^{1} (3) e^{-j \frac{2\pi}{5} n} = \begin{cases} \frac{5 \times (-2)}{5} + \frac{3 \times (3)}{5} & k = 5m \\ \frac{-2 e^{j \frac{2\pi}{5}} - e^{-j \frac{4\pi}{5}}}{5} & k = 5m \pm 1 \\ \frac{-2 e^{j \frac{4\pi}{5}} - e^{-j \frac{2\pi}{5}}}{5} & k = 5m \pm 2 \end{cases}$$

(c) On the graph below, plot the phase of $a_k$.

It is clear that:

$$\begin{cases} a_k < 0 \Rightarrow \angle a_k = \pm \pi & k = 5m \quad \forall m \in Z \\ a_k > 0 \Rightarrow \angle a_k = 0 & k = 5m \pm 1 \quad \forall m \in Z \\ a_k < 0 \Rightarrow \angle a_k = \pm \pi & k = 5m \pm 2 \quad \forall m \in Z \end{cases}$$

Therefore, it yields:

The following plot is also right:
Problem 4 (21 Points)

For each of the following systems, determine whether an LTI system exists that produces $y(t)$ for the input $x(t)$.

(a)  
\[ x(t) = 5 + \cos(\pi t + \frac{\pi}{4}) \quad \rightarrow \quad A \quad \rightarrow \quad y(t) = \cos^2\left(\frac{\pi}{2} t\right) \]

System A CAN be an LTI system.  
**Brief explanation:** The output of the system for the input  
\[ x(t) = 5 + \cos(\pi t + \frac{\pi}{4}) \]

is given by  
\[ y(t) = \cos^2\left(\frac{\pi}{2} t\right) = \frac{1}{2} + \frac{1}{2} \cos(\pi t) \]

where we’ve used the double angle formula. Therefore, the system can be LTI with frequency response that satisfies these conditions:  
\[ H(j\omega)|_{\omega=0} = \frac{1}{10} \]
\[ H(j\omega)|_{\omega=\pi} = H^*(j\omega)|_{\omega=-\pi} = \frac{1}{2} e^{-j\frac{\pi}{4}} \]

(b)  
\[ x(t) \quad \rightarrow \quad B \quad \rightarrow \quad y(t) \]

Where $x(t)$, $y(t)$ are periodic signals with fundamental periods $T$ and $\frac{T}{2}$, respectively.

System B CAN be an LTI system.  
**Brief explanation:** An example of an LTI system for which this condition is satisfied is  
\[ y(t) = x(t) + x \left( t - \frac{T}{2} \right) \]
(c) Where $x_1(t), x_2(t)$ and $y(t)$ are periodic signals with fundamental period $T$. The Fourier series coefficients $a_k$ of $x_1(t)$ and $b_k$ of $x_2(t)$ as well as one period of $y(t)$ are given below.
System C CAN be an LTI system.

**Brief explanation:** The only difference between \( a_k \) and \( b_k \) is for \( k = 0 \). Therefore, \( x_2(t) = x_1(t) - \frac{5}{8} \). In order for the system to be LTI it must completely attenuates the DC component \( (H(j\omega)|_{\omega=0} = 0) \), otherwise we couldn’t get the same output for both \( x_1(t) \) and \( x_2(t) \). Moreover, we have to show that a LTI system can produce the output \( y(t) \) for each one of the inputs. however, It is sufficient to show that \( y(t) \) can be produced for the zero DC signal \( x(t) = x_1(t) - \frac{1}{2} \). We know that if the system is LTI and the input is periodic, the relation between the output Fourier series (FS) coefficients and the input FS coefficients is given by

\[
a_y^k = a_x^k H(jk\omega_0)
\]

Now, the FS coefficients of \( x(t) \), \( a_x^k \) are all zero for even \( k \). Therefore, it follows from (2) that the system can be LTI only if the FS coefficients \( a_y^k \) are all zero for all even \( k \). i.e.,

\[
a_y^k = \begin{cases} c_k & \text{kodd} \\ 0 & \text{keven} \end{cases} = \frac{(1 - (-1)^k)}{2} \cdot c_k
\]

Transforming to the time domain, we get

\[
y(t) = \frac{(z(t) - z(t - \frac{T}{2}))}{2}
\]

and this can be accomplished by choosing \( z(t) < - > c_k \) to be the following periodic T signal
Problem 5  (26 Points)

Suppose we have an LTI system with an impulse response $h[n]$

(a) Find the frequency response $H(e^{j\omega})$ of this system.

$$H(e^{j\omega}) = \frac{1 + e^{-j\omega}}{2}$$

(b) Given the following input signal $x_b[n]$, find out the output signal $y_b[n]$.

$$x_b[n] = 1$$

$$y_b[n] = 1$$
Brief explanation:
There are two ways to solve this problem. One way is to convolve \( h[n] \) with \( x_b[n] \) directly and the result \( y_b[n] \) is 1. Another way is through system frequency response. The input signal’s frequency is DC and \( H(e^{j0}) = 1 \). Thus the system output is

\[
y_b[n] = H(e^{j0}) x_b[n] = 1
\]

(c) Given the following input signal \( x_c[n] \), find out the output signal \( y_c[n] \).

\[
x_c[n] = (-1)^n
\]

\[
y_c[n] = 0
\]

Brief explanation:
There are two ways to solve this problem too. One way is to convolve \( h[n] \) with \( x_c[n] \) directly and the result \( y_c[n] \) is 1. Another way is through system frequency response. The input signal’s frequency is \( \pi \) and \( H(e^{j\pi}) = 0 \). Thus the system output is

\[
y_c[n] = H(e^{j\pi}) x_c[n] = 0
\]

(d) Consider the following system:

Given the following input signal \( x[n] \):

Evaluate this expression: \( \lim_{M \to +\infty} y_M[n] \).
\[ \lim_{M \to +\infty} y_M[n] = 5 \]

**Brief explanation:** It is clear that

\[ x[n] = 5x_b[n] + 2x_c[n] \]

Thus,

\[ y_1[n] = 5y_b[n] + 2y_c[n] = 5 \]

Furthermore

\[ y_k[n] = \cdots = y_2[n] = y_1[n] = 5 \]

Therefore

\[ \lim_{M \to +\infty} y_M[n] = 5 \]

(e) For a DT periodic signal \( x[n] \) with **arbitrary** period \( N \), evaluate \( \lim_{M \to +\infty} y_M[n] \).

\[ \lim_{M \to +\infty} y_M[n] = \frac{1}{N} \sum_{<N>} x[n] \]

**Reason:** The frequency response of the system with input \( x[n] \) and output \( y_M[n] \) is:

\[ H_M(e^{j\omega}) = \left( \frac{1 + e^{-j\omega}}{2} \right)^M \]

Furthermore,

\[ \lim_{M \to \infty} H_M(e^{j\omega}) = \lim_{M \to \infty} \left( \frac{1 + e^{-j\omega}}{2} \right)^M = \begin{cases} 1 & \omega = 2k\pi \\ 0 & \text{otherwise} \end{cases} \]

Thus, as \( M \to \infty \), only the DC signal pass through and all the other frequency signals are attenuated completely. Thus,

\[ \lim_{M \to +\infty} y_M[n] = \frac{1}{N} \sum_{<N>} x[n] \]