Directions:

- Please make sure your name is on all sheets.
- Unless indicated otherwise, answers must be derived or briefly explained. All sketches must be adequately labeled.
- Enter all your work and answers directly in the spaces provided on the printed pages of this booklet. Additional work spaces are supplied from Pages 24 through 26. These pages will not be graded unless you specify that you are continuing a particular problem on a particular continuation page.
- This Quiz is closed book except for two 8 1/2 × 11-inch sheet of paper (four sides).
- Tables of Fourier series and transform properties and pairs are a separate handout. You must turn in your transform sheets when you turn in your quiz.
- The bluebook is for your scratch work. We will not grade anything in your bluebook.
- No other materials and aids, such as calculators, cell phones, or music players, are permitted.

NAME: SOLUTIONS

<table>
<thead>
<tr>
<th>Check Your Section</th>
<th>Section</th>
<th>Time</th>
<th>Room</th>
<th>Rec. Instr.</th>
<th>TA</th>
</tr>
</thead>
<tbody>
<tr>
<td>☐</td>
<td>1</td>
<td>10–11</td>
<td>34-301</td>
<td>Prof. Joel Voldman</td>
<td>Saikat Guha</td>
</tr>
<tr>
<td>☐</td>
<td>2</td>
<td>11–12</td>
<td>34-301</td>
<td>Prof. Joel Voldman</td>
<td>Saikat Guha</td>
</tr>
<tr>
<td>☐</td>
<td>3</td>
<td>12–1</td>
<td>34-301</td>
<td>Dr. Karen Livescu</td>
<td>Kevin Boyle</td>
</tr>
<tr>
<td>☐</td>
<td>4</td>
<td>1–2</td>
<td>34-301</td>
<td>Dr. Karen Livescu</td>
<td>Kevin Boyle</td>
</tr>
<tr>
<td>☐</td>
<td>5</td>
<td>1–2</td>
<td>24-407</td>
<td>Prof. Russ Tedrake</td>
<td>Kushan Surana</td>
</tr>
<tr>
<td>☐</td>
<td>6</td>
<td>3–4</td>
<td>34-301</td>
<td>Stephen Hou</td>
<td>Chun-Chieh Lin</td>
</tr>
</tbody>
</table>

Please leave the rest of this page blank for use by the graders:

<table>
<thead>
<tr>
<th>Problem</th>
<th>No. of points</th>
<th>Score</th>
<th>Grader</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>19</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>22</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>20</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>21</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>18</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>100</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
PROBLEM 1 (19%)

For a signal $x(t)$, the magnitude and phase of its Fourier transform, $X(j\omega)$, are given below.

A. Compute $\int_{-\infty}^{\infty} x(t) dt$ and $\int_{-\infty}^{\infty} |x^2(t)| dt$:

$$\int_{-\infty}^{\infty} x(t) dt = \boxed{2}$$

Solution:
The analysis equation is $X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$. Setting $\omega = 0$ produces $X(0) = \int_{-\infty}^{\infty} x(t) dt$.

From the graph of $X(j\omega)$ above, we have $|X(0)| = 2$ and $\angle X(0) = 0$. Thus, $\int_{-\infty}^{\infty} x(t) dt = X(0) = |X(0)|e^{j\angle X(0)} = 2e^{j0} = 2$.

$$\int_{-\infty}^{\infty} |x^2(t)| dt = \frac{140}{3\pi}$$

Solution:

(See next page)
Work page for problem 1

Solution:
From Parseval’s Relation, we have \( \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 \, d\omega \). The equation of the magnitude of \( X(j\omega) \) for \( 0 < \omega < 5 \) is \( |X(j\omega)| = \frac{4}{5} \omega + 2 \). Thus, using symmetry,
\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 \, d\omega = \frac{1}{2\pi} \int_{0}^{5} \left( \frac{4}{5} \omega + 2 \right)^2 \, d\omega = \frac{4}{25} \omega^2 + \frac{8}{5} \omega + 4 \\int_{0}^{5} d\omega = \frac{1}{\pi} \left[ \frac{4}{3} 5^3 + \frac{4}{5} 5^2 + 4 \cdot 5 \right] = \frac{1}{\pi} \left[ \frac{20}{3} + 20 + 20 \right] = \frac{140}{3\pi}.
\]

You can also use a clever series of change of variables to simplify the integral if you are prone to making arithmetic errors:
\[
\frac{1}{2\pi} \int_{0}^{5} \left( \frac{2}{5} \omega + 2 \right)^2 \, d\omega = \frac{1}{\pi} \int_{0}^{5} \left( \frac{1}{5} \omega + 1 \right)^2 \, d\omega \quad \text{(Pulled out a factor of 4)}.
\]
\[
= \frac{1}{\pi} 4 \cdot 5 \int_{0}^{1} (u + 1)^2 \, du, \quad \text{where } u = \omega/5, d\omega = 5 \, du.
\]
\[
= \frac{1}{\pi} 4 \cdot 5 \int_{1}^{2} x^2 \, dx, \quad \text{where } x = u + 1, du = dx.
\]
\[
= \frac{1}{\pi} 4 \cdot 5 \left[ \frac{1}{3} x^3 \right]_{1}^{2} = \frac{1}{\pi} 4 \cdot 5 \left[ \frac{8 - 1}{3} \right] = \frac{1}{\pi} 4 \cdot 5 \cdot \frac{7}{3} = \frac{140}{3\pi}.
\]

Common Errors:
Many people forgot to square the magnitude and found the area of the magnitude of the Fourier transform instead. This produces an answer of \( \frac{30^2}{2\pi} = \frac{15}{\pi} \). Other students squared the area afterwards to get \( \frac{30^2}{2\pi} = \frac{450}{\pi} \).

Another common mistake was forgetting that squaring a linear function produces a quadratic. Some students simply squared the heights at \( \omega = 0 \) (to get 4) and \( \omega = 5 \) (to get 16) and used it as a triangle. This produces an answer of \( \frac{100}{2\pi} = \frac{50}{\pi} \).
**B.** For each of the signals below, find the matching magnitude and phase plots from the ones on the following two pages. **No explanation necessary. No partial credit.**

<table>
<thead>
<tr>
<th>Signal</th>
<th>Magnitude Plot</th>
<th>Phase Plot</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x(-t)$</td>
<td>A</td>
<td>III</td>
</tr>
<tr>
<td>$\frac{dx(t)}{dt}$</td>
<td>F</td>
<td>II</td>
</tr>
<tr>
<td>$x(t)e^{j2t}$</td>
<td>H</td>
<td>I</td>
</tr>
<tr>
<td>$x(t) * \frac{\sin(3t)}{\pi t}$</td>
<td>B</td>
<td>V</td>
</tr>
<tr>
<td>$x(t)[\delta(t) + \delta(t - 8)]$</td>
<td>E</td>
<td>VII</td>
</tr>
<tr>
<td>$x(\cos(t))$</td>
<td>D</td>
<td>xxx</td>
</tr>
<tr>
<td>$x(t - \frac{\pi}{10})$</td>
<td>A</td>
<td>VI</td>
</tr>
</tbody>
</table>

†: Please note that this function is $x(\cos(t))$, not $x(t)\cos(t)$. We are not asking you to choose a phase plot for this signal.

**Magnitude plots for Part B.**

![Magnitude Plots](image-url)
Phase plots for Part B.

Solution:

This part tests your conceptual understanding of Fourier transform properties.

- $x(-t)$ is a time reversal, which corresponds to a frequency reversal. So, magnitude and phase are both flipped across the vertical axis.
- $\frac{dx(t)}{dt}$ is differentiation, which corresponds to multiplication by $j\omega$. This means that the magnitude is multiplied by $\omega$ and the phase is shifted by $\pi/2$ (the phase of $j$).
- $x(t)e^{j2t}$ corresponds to a frequency shift by 2 (modulation).
- $x(t)\frac{\sin(3t)}{\pi t}$ corresponds to lowpass filtering.
- $x(t)[\delta(t) + \delta(t - 8)]$ is a “discrete” signal in time, so it is periodic in frequency.
- $x(\cos(t))$ is periodic because $\cos(t)$ is periodic. Thus, the Fourier transform is discrete (consists of delta functions).
- $x(t - \frac{\pi}{10})$ is a time shift, which corresponds to the addition of a linear phase. Thus, the magnitude stays the same. In this case, the added linear phase exactly cancels the original phase in $x(t)$, so the new phase is zero.
PROBLEM 2 (22%)

In this problem we will consider anti-causal DT LTI filters.

A. Write down the linear, constant coefficient difference equation (LCCDE) for the system with unit sample (impulse) response as given by \( h_1[n] \).

Solution:
The impulse response is \( h_1[n] = 0.2(\delta[n] + \delta[n + 1] + \delta[n + 2] + \delta[n + 3] + \delta[n + 4]) \). Convolving it with \( x[n] \) produces the LCCDE as given below.

\[
LCCDE: \quad y[n] = 0.2(x[n] + x[n + 1] + x[n + 2] + x[n + 3] + x[n + 4])
\]

B. Derive the frequency response of the system described by \( h_1[n] \).

Solution:
We take the Fourier transform of \( h_1[n] \) to get the frequency response.

\[
H_1(e^{j\omega}) = 0.2(1 + e^{j\omega} + e^{2j\omega} + e^{3j\omega} + e^{4j\omega})
\]
C. How does the magnitude and phase of this frequency response differ from the magnitude and phase of the frequency response of the causal DT filter described by $h_2[n]$ (shown below)?

Circle one of the following. No explanation necessary. No partial credit.

1. $|H_1(e^{j\omega})| = |H_2(e^{j\omega})|$
2. $|H_1(e^{j\omega})| = 2|H_2(e^{j\omega})|$
3. $|H_1(e^{j\omega})| = |H_2(e^{j\omega})| + 1$
4. $|H_1(e^{j\omega})| = \frac{1}{|H_2(e^{j\omega})|}$

Circle one of the following. No explanation necessary. No partial credit.

1. $\angle H_1(e^{j\omega}) = \angle H_2(e^{j\omega})$
2. $\angle H_1(e^{j\omega}) = 2\angle H_2(e^{j\omega})$
3. $\angle H_1(e^{j\omega}) = -\angle H_2(e^{j\omega})$
4. $\angle H_1(e^{j\omega}) = \angle H_2(e^{j\omega}) + \pi/2$

Solution:
There are two ways of thinking about $h_2[n]$ in terms of $h_1[n]$: $h_2[n] = h_1[n - 8] = h_1[-n]$, i.e. $h_2[n]$ is $h_1[n]$ delayed or time reversed. The first interpretation tells us that the magnitudes of the Fourier transforms are the same (Choice 1 for magnitude) because this is preserved under a time delay. The second interpretation tells us that the phases of the Fourier transforms are opposites (Choice 3 for phase) because time reversal flips phase around the vertical axis. Since phase is an odd function, this is equivalent to flipping it around the horizontal axis.
D. Consider the DT LTI filter described by the unit sample (impulse) response:

\[ h_3[n] = \begin{cases} (-\alpha)^{-n} & n \leq 0 \\ 0 & n > 0 \end{cases} \]

where \(0 < \alpha < 1\). Write the linear constant coefficient difference equation that describes this system without using infinite sums.

**Solution:**
It is easier to do the next part first! From the frequency response we get:

\[ \frac{Y(e^{j\omega})}{X(e^{j\omega})} = H_3(e^{j\omega}) = \frac{1}{1 + \alpha e^{j\omega}}. \]

So, \(Y(e^{j\omega}) + \alpha e^{j\omega}Y(e^{j\omega}) = X(e^{j\omega})\). Taking the inverse Fourier transform yields the LCCDE.

**LCCDE:** \[ y[n] + \alpha y[n+1] = x[n] \]

E. Derive the frequency response for the system described by \(h_3[n]\).

**Solution:**
\(h_3[n]\) is the time-reversed version of \((-\alpha)^nu[n]\), which has Fourier transform \(\frac{1}{1 + \alpha e^{-j\omega}}\). Time-reversal corresponds to frequency-reversal, so the frequency response is:

\[ H_3(e^{j\omega}) = \frac{1}{1 + \alpha e^{j\omega}} \]

F. Which type of filter is implemented by the system \(h_3[n]\)? **Circle one of the following. No explanation necessary. No partial credit.**

1. Low pass
2. **High pass**
3. Band pass

**Solution:**
When \(\omega = 0\) (low frequency), the frequency response is \(\frac{1}{1+\alpha}\). When \(\omega = \pi\) (high frequency), the frequency response is \(\frac{1}{1-\alpha}\). Since \(0 < \alpha < 1\), this means that the system passes high frequencies more than it does low ones. Furthermore, the magnitude of the frequency response is a monotonic function as frequency rises from 0 to \(\pi\). Thus, the system is a **high pass** filter.
Work page for problem 2
PROBLEM 3 (20%)

Consider the following CT system:

\[ \begin{align*}
    x(t) & \quad H_c(j\omega) & \quad y(t) \\
    -10\pi & \quad 0 & \quad 10\pi \\
\end{align*} \]

and the following DT processing system:

\[ \begin{align*}
    x(t) & \quad \times_{\underline{x}} \quad p(t) \quad T_s \quad \text{Impulse to Sequence} \quad T_s \quad \underline{x}_d[n] \\
    \quad & \quad H_d(e^{j\Omega}) \quad y_d[n] \quad \text{Sequence to Impulse} \\
    \quad & \quad T_s \quad y_p(t) \quad \omega \\
\end{align*} \]

where \( p(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT_s) \). We would like to design \( H_d(e^{j\Omega}) \) to perform the same processing as \( H_c(j\omega) \) for the input

\[ x(t) = \frac{\sin(50\pi t)}{\pi t}. \]
A. Sketch $Y(j\omega)$, the output of the CT system for the input described above. Label your $\omega$ and $Y(j\omega)$ axes with values.

**Solution:**

Note that $X(j\omega) = \begin{cases} 1 & |\omega| < 50\pi \\ 0 & |\omega| > 50\pi \end{cases}$

This gives $Y(j\omega) = X(j\omega)H_c(j\omega) = \begin{cases} 1 & 10\pi < |\omega| < 50\pi \\ 0 & \text{otherwise} \end{cases}$

B. Sketch $X_d(e^{j\Omega})$, the DTFT of $x_d[n]$, for $T_s = \frac{1}{100}$. Label your axes with values.

**Solution:**

First note that the Nyquist sampling condition is satisfied ($\omega_s = \frac{2\pi}{T_s} > 2(50\pi)$).

Therefore, $X_d(e^{j\Omega}) = \frac{1}{T_s}X(j\frac{\Omega}{T_s})$ for $|\Omega| < \pi$, and it is periodic with period $2\pi$. 

---

11
C. Design $H_d(e^{j\Omega})$ as an ideal DT high-pass filter so that the complete DT processing system implements $H_c(j\omega)$ [i.e., $y_c(t) = y(t)$ for this $x(t)$] for $T_s = \frac{1}{100}$. Sketch your $H_d(e^{j\Omega})$ on the axes below. Label your axes with values.

Solution:
From part A, we know that this filter needs to filter out the middle $\frac{1}{5}$ of the “box” in the $X(j\omega)$, while leaving the rest of it intact. Since the scaling is all linear, this means that $H_d(e^{j\Omega})$ must filter out the middle $\frac{1}{5}$ of each “box” in the $X_d(e^{j\Omega})$, which we have from part B. There is only one way to do this, since $H_d(e^{j\Omega})$ has to be an ideal DT high-pass filter.

\[ H_d(e^{j\Omega}) \]

\[ \Omega \]

D. What happens if you use the exact $H_d(e^{j\Omega})$ that you designed in part C with $T_s = \frac{1}{50}$? Sketch and label the output, $Y_c(j\omega)$, of the DT processed signal with this new sampling rate with the specified $x(t)$.

Solution:
First note that we are sampling at (exactly) the Nyquist rate ($\omega_s = \frac{2\pi}{T_s} = 2(50\pi)$). This does not introduce aliasing issues for this input though, since $X(j\omega)$ contains no impulses. However, each “box” in the $X_d(e^{j\Omega})$ is now twice as wide, so the $H_d(e^{j\Omega})$ from part C only filters out the middle $\frac{1}{10}$ of each “box” in $X_d(e^{j\Omega})$. This means that only the middle $\frac{1}{10}$ of the “box” in $X(j\omega)$ is filtered out.

\[ Y_c(j\omega) \]

\[ \omega \]
E. What is the upper limit, if one exists, of $T_s$ for which you can construct an **ideal DT high-pass filter** for $H_d(e^{j\Omega})$ so that $y_c(t) = y(t)$ for this $x(t)$? **Write a value or write “No Solution”**.

**Solution:**
This is basically asking for the Nyquist sampling rate. If we sample at less than the Nyquist rate, then aliasing would prevent us from meeting the condition. Like we mentioned in part D, $\frac{1}{50}$ is the Nyquist rate for our input signal.

$$T_s < \frac{1}{50}$$

F. What is the upper limit, if one exists, of $T_s$ for which you can construct an **ideal DT high-pass filter** for $H_d(e^{j\Omega})$ so that $y_c(t) = y(t)$ if the input, $x(t)$, is changed to the signal drawn here:

![Signal Diagram](image)

**Write a value or write “No Solution”**.

**Solution:**
Note that the transform of this signal is a sinc function, which is not band-limited. Therefore, no $T_s$ would work.

$$T_s < \text{No Solution}$$
Work page for problem 3
PROBLEM 4 (21%)

Consider the following CT LTI system

\[ H(j\omega) = \frac{-(100+j\omega)}{10(1+j\omega)} \]

A. Determine \( h(t) \), the impulse response of the system.

**Solution:**

\[ H(j\omega) = \frac{-(100 + j\omega)}{10(1 + j\omega)} = -\frac{1}{10} \left( \frac{99}{1 + j\omega} + \frac{1 + j\omega}{1 + j\omega} \right) = -\frac{1}{10} \left( \frac{99}{1 + j\omega} + 1 \right). \]

Taking the inverse Fourier transforms yields the impulse response.

\[ h(t) = \frac{-99}{10} e^{-t} u(t) - \frac{1}{10} \delta(t) \]

B. Determine a linear constant coefficient differential equation (LCCDE) described by this frequency response.

**Solution:**

\[ \frac{Y(j\omega)}{X(j\omega)} = H(j\omega) = \frac{-(100 + j\omega)}{10(1 + j\omega)} = \frac{-100 - j\omega}{10 + 10j\omega}. \] Thus,

\[ 10Y(j\omega) + 10j\omega Y(j\omega) = -100X(j\omega) - j\omega X(j\omega). \]

Taking the inverse Fourier transform yields the LCCDE.

\[ \text{LCCDE:} \quad 10y(t) + 10 \frac{d}{dt} y(t) = -100x(t) - \frac{d}{dt} x(t) \]
C. Sketch approximate Bode plots of the magnitude and phase responses of this system, $|H(j\omega)|$ and $\angle H(j\omega)$, on the axes below. Only straight-line approximations are needed. Label all important features (e.g. slope of each line and points where lines intersect).
D. Determine $y(t)$ when $x(t) = 3e^{-t}u(t)$.

**Solution:**
We take the Fourier transform $X(j\omega)$ of $x(t)$:

$$X(j\omega) = \frac{3}{1 + j\omega}.$$  

We multiply it by the frequency response $H(j\omega)$ to get $Y(j\omega)$:

$$Y(j\omega) = X(j\omega)H(j\omega) = \frac{-\frac{3}{10}(100 + j\omega)}{(1 + j\omega)^2} = \frac{A}{(1 + j\omega)^2} + \frac{B}{1 + j\omega}. $$

So, $A + B + Bj\omega = -30 - \frac{3}{10}j\omega$. This means that $B = -0.3$ and $A = -29.7$:

$$Y(j\omega) = \frac{-29.7}{(1 + j\omega)^2} + \frac{-0.3}{1 + j\omega}.$$  

Taking the inverse Fourier transform yields the output.

$$y(t) = [-29.7t - 0.3]e^{-t}u(t)$$

E. Determine $y(t)$ when $x(t) = 4\cos\left(50t + \frac{\pi}{4}\right)$.

**Solution:**
(See next page)

$$y(t) = 20\sqrt{\frac{5}{25001}} \cos \left(50t + \frac{5\pi}{4} + \arctan \left(\frac{1}{2}\right) - \arctan(50)\right) \approx \frac{4}{5} \cos \left(50t + \frac{37\pi}{40}\right)$$
Work page for problem 4

Solution:
Use the eigenfunction property! The output is scaled by the magnitude of the frequency response (evaluated at $\omega = 50$) and shifted by the phase (likewise): $y(t) = 4|H(j50)|\cos(50t + \frac{\pi}{4} + \angle H(j50))$.

$$H(j50) = \frac{-100 - j50}{10 + 500j} = \frac{-10 - j5}{1 + 50j}.$$  Thus,

$$|H(j50)| = \sqrt{H(j50)H^*(j50)} = \sqrt{\left(\frac{-10 - j5}{1 + 50j}\right)\left(\frac{-10 + j5}{1 - 50j}\right)} = \sqrt{\frac{100 + 25}{1 + 2500}} = \sqrt{\frac{100}{2501}} = \frac{\sqrt{5}}{2501}.$$  

$$\angle H(j50) = \angle(-100 - j50) - \angle(10 + 500j) = \pi + \arctan\left(\frac{1}{2}\right) - \arctan(50).$$

So:

$$y(t) = 20\sqrt{\frac{5}{2501}}\cos\left(50t + \frac{\pi}{4} + \arctan\left(\frac{1}{2}\right) - \arctan(50)\right).$$

Whew! That was quite a bit of math. Is there another way? To pose the question is to know the answer! Since we already have the Bode plot from Part C, we can get a good estimate of the magnitude and phase of $H(j50)$ extremely elegantly. For $10 < \omega < 100$, the magnitude of $H(\omega)$ is approximated by a straight line on a log-log scale with slope $-20$ dB/decade. This means that if $\omega$ scales up by some factor $\alpha$, magnitude will scale down by $\alpha$. $|H(\omega)|$ at $\omega = 10$ is 1 (or 0 dB) so when $\omega = 50$ (scale up by a factor of 5), $|H(j50)|$ is about 0.2 (scales down by a factor of 5). So, $|H(j50)| \approx 0.2$.

Similarly, the phase of $H(\omega)$ is approximately linear, though on a linear-log scale. We have two points on this line: $\angle|H(j10)| = \frac{\pi}{2}$ and $\angle|H(j100)| = \frac{3\pi}{4}$. Now we need to know where $\omega = 50$ fits in. Since $\log 10 = 1$, $\log 100 = 2$, and $\log 50 = 1 + \log 0.5 \approx 1.7$ (you should know this approximation!), the phase at $\omega = 50$ is 0.7 of the way from $\frac{\pi}{2}$ to $\frac{3\pi}{4}$, or $\pi\left(\frac{1}{2} + 0.7\frac{1}{4}\right) = \frac{27\pi}{40}$. So, $\angle H(j50) \approx \frac{27\pi}{40} = 0.675\pi$.

Thus, an approximation for the output is

$$y(t) \approx \frac{1}{5}4\cos\left(50t + \frac{1\pi}{4} + \frac{27\pi}{40}\right) = \frac{4}{5}\cos\left(50t + \frac{37\pi}{40}\right).$$

How good is this approximation? $5\sqrt{\frac{5}{2501}} = 0.22356...$, which is close enough to 0.2. Similarly, $\pi + \arctan\left(\frac{1}{2}\right) - \arctan(50) = \pi - 0.65394...$, which is close enough to $0.675\pi$. This problem shows how the Bode plot enables us to get a quick, intuitive feel for the output of a system when the input is a pure sinusoid.
PROBLEM 5 (18%)

Consider a DT signal $x[n]$, about which we have the following information:

- $x[n]$ is real and even
- $X(e^{j\omega}) = 0$ for $\omega_M \leq |\omega| \leq \pi$

A. Which of the following graphs could represent $X(e^{j\omega})$? Circle the letters for all possible graphs. No explanation is necessary. No partial credit.

Solution:
Since $x[n]$ is real and even, $X(e^{j\omega})$ is also real and even. Furthermore, $X(e^{j\omega})$ is 0 for $\omega_M \leq |\omega| \leq \pi$. Only B and D satisfy all these conditions.
B. We are given this signal with one sample corrupted, i.e. we are actually given

\[ x_1[n] = \begin{cases} x[n] & \text{for all } n \neq n_0 \\ x[n_0] + C & \text{if } n = n_0 \end{cases} \]

for some real \( C \neq 0 \) and integer \( n_0 \). We are also given \( X_1(e^{j\omega}) \), the DTFT of \( x_1[n] \). Our ultimate goal is to restore the original signal, i.e. to determine \( n_0 \) and \( x[n_0] \). We will do this in several steps.

(i) First, write a closed-form expression for \( x_1[n] \) in terms of \( x[n] \) and a unit sample (impulse) function. **Solution:** \( x_1[n] \) is \( x[n] \) modified only at \( n = n_0 \), where the signal is changed by \( C \).

\[ x_1[n] = x[n] + C\delta[n - n_0] \]

(ii) Next, write \( X_1(e^{j\omega}) \) in terms of \( X(e^{j\omega}) \) and the quantities \( \omega_M, n_0, \) and \( C \). Also plot \( |X_1(e^{j\omega})| \) and \( \angle X_1(e^{j\omega}) \) only for \( \omega_M < |\omega| < \pi \) on the axes below. Fully label your plot. **Solution:** Take the Fourier transform of the answer from the previous part.

\[ X_1(e^{j\omega}) = X(e^{j\omega}) + Ce^{-j\omega n_0} \]

\[ |X_1(e^{j\omega})| \]
Solution:

Since $|X(e^{j\omega})|$ is zero for the range specified, $|X_1(e^{j\omega})| = |Ce^{-j\omega_0}| = |C|$ and $\angle X_1(e^{j\omega}) = \angle C - \omega n_0$ for $\omega_M < |\omega| < \pi$. Note that $C$ may be either positive or negative. For the phase, this means that $\angle X_1(e^{j\omega})$ is $-\omega n_0$ when $C$ is positive and is $-\omega n_0 \pm \pi$ when $C$ is negative.

Common Errors:

By far the most common error was not realizing that $C$ may be either positive or negative. Another error was plotting $|X_1(e^{j\omega})|$ for the wrong range. Please follow directions!
(iii) Give a valid expression for $C$ in terms of $X_1(e^{j\omega})$ and $\omega_M$.

**Solution:**
This comes from part (ii). The first condition is when $C$ is positive; the second is when $C$ is negative.

$$C = \begin{cases} |X_1(e^{j\omega_M})|, & \text{if } \frac{d}{d\omega} \angle X_1(e^{j\omega}) = \angle X_1(e^{j\omega})/\omega \text{ for } \omega_M < \omega < \pi. \\ -|X_1(e^{j\omega_M})|, & \text{else} \end{cases}$$

(iv) Give a valid expression for $n_0$ in terms of $X_1(e^{j\omega})$ and $\omega_M$.

**Solution:**
The negative slope of the phase is $n_0$. Note that this expression takes care of both positive and negative $C$. Using $-\angle X_1(e^{j\omega})/\omega$ only works for positive $C$.

$$n_0 = -\frac{d}{d\omega} \angle X_1(e^{j\omega})|_{\omega\in(\omega_M,\pi)}$$

(v) Therefore, what is $x[n_0]$, in terms of the received signal $x_1[n]$ and its Fourier transform $X_1(e^{j\omega})$?

**Solution:**
From the problem statement, we see that $x[n_0] = x_1[n_0] - C$. Thus, we get:

$$x[n_0] = x_1 \begin{cases} -\frac{d}{d\omega} \angle X_1(e^{j\omega})|_{\omega\in(\omega_M,\pi)} & \text{if } \frac{d}{d\omega} \angle X_1(e^{j\omega}) = \angle X_1(e^{j\omega})/\omega \text{ for } \omega_M < \omega < \pi. \\ |X_1(e^{j\omega_M})|, & \text{else} \end{cases}$$
Work Page for Problem 5
Additional Work Page
Additional Work Page
Additional Work Page