Directions: The exam consists of six problems on pages 2 to 27. Please make sure you have all the pages. Tables are supplied to you at the end of this booklet. Enter all your work and your answers directly in the spaces provided on the printed pages of this booklet. Please make sure your name is on all sheets. DO IT NOW! All sketches must be adequately labeled. This examination is closed book, but students may use two 8 1/2 x 11 sheets of paper for reference. All electronic devices (including phones, pagers, calculators, computers, cameras, and music players) must be turned off and stowed completely out of sight. A simple timepiece (such as a watch) may be permitted at the discretion of the teaching staff.

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PROBLEM 1 (18%)

Part a. Consider the following CT system.

\[ e^{-t}u(t) \rightarrow \delta(t) \rightarrow e^{-2t}u(t) \rightarrow g(t) \rightarrow e^{-t}u(t) \rightarrow te^{-t}u(t) \]

The boxes represent LTI systems, and the enclosed expressions are their impulse responses. Determine \( g(t) \).

\[ e^{-t}u(t) \leftrightarrow \frac{1}{1+jw} \]
\[ e^{-2t}u(t) \leftrightarrow \frac{1}{2+jw} \]
\[ t e^{-t}u(t) \leftrightarrow \frac{1}{(1+jw)^2} \]

\[ \left[ \frac{1}{1+jw} + 1 \cdot G(jw) \right] \frac{1}{2+jw} = \frac{1}{(1+jw)^2} \]

\[ G(jw) = \frac{2+jw}{(1+jw)^2} - \frac{1}{1+jw} \]
\[ = \frac{1}{(1+jw)^2} \]

\[ g(t) = te^{-t}u(t) \]
Spring 2004: Quiz 2

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Work page for Problem 1
Part b. Consider the following DT system.

\[ x[n] \xrightarrow{} h[n] \xrightarrow{} y[n] \]

\[ x[n] = \left[ \sin \left( \frac{\pi}{\pi} n \right) \right]^2 \]

\[ h[n] = \frac{\sin[\frac{\pi}{2}(n - 1)]}{\pi(n - 1)} \]

The box represents an LTI system with impulse response \( h[n] \). Sketch the magnitude and phase of \( Y(e^{j\omega}) \), which is the Fourier transform of \( y[n] \), between \(-2\pi \) and \( 2\pi \) on the axes below. Carefully label all important features (including heights on the vertical axes).
Work page for Problem 1

\[ X[n] = \frac{\sin \frac{\pi}{2} n}{\pi n} \cdot \frac{\sin \frac{\pi}{2} n}{\pi n} \]

\[ \frac{1}{2\pi} \quad \star \quad \frac{1}{2\pi} \]

\[ h'[n] = \frac{\sin \frac{\pi}{2} n}{\pi n} \]

\[ |H(e^{j\omega})| \]

\[ h[n] = h[n-1] \]

\[ H(e^{j\omega}) = H'(e^{j\omega}) e^{-j\omega} \]

Linear phase

\[ Y(e^{j\omega}) = X(e^{j\omega}) H(e^{j\omega}) \]

\[ |Y(e^{j\omega})| = |X(e^{j\omega})| |H(e^{j\omega})| \]

\[ \tilde{Y}(e^{j\omega}) = \tilde{X}(e^{j\omega}) + \frac{\tilde{X}}{H(e^{j\omega})} \]

\[ \tilde{Y}(e^{j\omega}) = \tilde{X}(e^{j\omega}) + \tilde{X} \cdot \tilde{H}(e^{j\omega}) \]
PROBLEM 2 (18%)

Part a. Consider the frequency response

\[ H(j\omega) = \frac{(j\omega - 1)(j\omega + 100)}{(j\omega + 10)(j\omega + 1000)} \]

Sketch an approximation for the magnitude and phase of \( H(j\omega) \) on the axes below. Carefully label all important features.
Magnitude:

\[ |H(j\omega)| = \frac{(j\omega)(100)}{(j\omega)(1000)} = \frac{1}{100} \Rightarrow \text{Bode plot starts off at } 20\log\left(\frac{1}{100}\right) = -40 \text{ dB} \]

\[ H(j\omega) = \frac{(j\omega-1)(j\omega+100)}{(j\omega+10)(j\omega+1000)} \]

up 20 dB/decade starting at \(\omega=1\)

\[ \text{up } 20 \text{dB/decade starting at } \omega=100 \]

down 20 dB/decade starting at \(\omega=10\)

\[ \text{down } 20 \text{dB/decade starting at } \omega=1000 \]

Phase:

\[ \angle H(j\omega) = \angle(-1) + \angle(100) - \angle(10) - \angle(1000) \]

\[ = \pi + 0 - 0 - 0 = \pi \]

\[ \Rightarrow \text{Bode phase plot starts off at } \pi \]

\[ H(j\omega) = \frac{(j\omega-1)(j\omega+100)}{(j\omega+10)(j\omega+1000)} \]

contributes \(-\pi/2\) over \(\omega=1\) to \(\omega=10\)

\[ \text{contributes } +\pi/2 \text{ over } \omega=10 \text{ to } \omega=1000 \]

\[ \text{contributes } -\pi/2 \text{ over } \omega=100 \text{ to } \omega=10000 \]
Part b. Consider the frequency response

\[ H(j\omega) = \frac{j\omega}{(j\omega)^2 + 0.01(j\omega) + 1}. \]

Sketch an approximation for the magnitude of \( H(j\omega) \) on the axes below. Carefully label all important features (including the approximate location and height of any resonance).

Determine the approximate values of the phase of \( H(j\omega) \) at \( \omega = 0.01 \) and \( \omega = 100 \).

\[ \angle H(j\omega) \big|_{\omega=0.01} = \frac{\pi}{2} \]

\[ \angle H(j\omega) \big|_{\omega=100} = -\frac{\pi}{2} \]

At \( \omega = 0.01 \), a decade before \( \omega_n \), the phase will only contain the contribution from the numerator, which is always \( \frac{\pi}{2} \).

At \( \omega = 100 \), a decade after \( \omega_n \), the phase will be the difference of the numerator phase (\( \frac{\pi}{2} \)) and the denominator phase (which is now \( \pi \) since we are at a decade above \( \omega_n \)):

\[ \frac{\pi}{2} - \pi = -\frac{\pi}{2} \]
Work page for Problem 2

We can split this up into two systems, analyze them separately, and sum their Bode plots to get the overall system Bode plot.

\[ H(j\omega) = H_1(j\omega) \cdot H_2(j\omega) \quad \text{where} \quad H_1(j\omega) = j\omega \]

\[ \text{and} \quad H_2(j\omega) = \frac{1}{(j\omega)^2 + 0.01(j\omega) + 1} \]

\[ 20 \log |H(j\omega)| \]

\[ \begin{array}{c}
\text{0 dB} \\
\text{20 dB/decade}
\end{array} \]

To analyze \( H_2(j\omega) \), we can compare the denominator to \((j\omega)^2 + 2\frac{\omega_n}{\zeta} j\omega + \omega_n^2\), and we can see that \( \omega_n = 1 \) & \( \frac{\omega_n}{\zeta} = 0.005 \)

Thus, the Bode plot will be constant at \( 20 \log (1) = 0 \text{ dB} \) until \( \omega_n = 1 \), at which point it will have a peak of height \( \frac{2\zeta}{\omega_n} = 100 \) dB, and then begin sloping downward at a slope of \(-40 \text{ dB/decade}\)

Summing the two Bode plots, we obtain:

\[ 20 \log |H(j\omega)| \]

\[ \begin{array}{c}
\text{40 dB} \\
\text{-40 dB/decade}
\end{array} \]
PROBLEM 3 (18%)

Part a. A causal and stable signal $x(t)$ has the Fourier transform

$$X(j\omega) = \frac{j\omega}{(j\omega + 1)(j\omega + 2)}.$$ 

Determine $\frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 d\omega$. 

$\frac{1}{6}$
Work page for Problem 3

By Parseval's Relation, we know that
\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} |X(j\omega)|^2 \, d\omega = \int_{-\infty}^{\infty} |x(t)|^2 \, dt
\]

We can determine \( x(t) \) and calculate this integral:

\[
X(j\omega) = \frac{j\omega}{(j\omega+1)(j\omega+2)} = \frac{A}{(j\omega+1)} + \frac{B}{(j\omega+2)}
\]

\[
A(j\omega+2) + B(j\omega+1) = j\omega
\]

\[\Rightarrow A = -1 \quad \text{and} \quad B = -2\]

\[\Rightarrow X(j\omega) = \frac{-1}{(j\omega+1)} + \frac{2}{(j\omega+2)}\]

\[\Rightarrow x(t) = -e^{-t}u(t) + 2e^{-2t}u(t)\]

Now to determine the integral:

\[
\int_{-\infty}^{\infty} |x(t)|^2 \, dt = \int_{-\infty}^{\infty} (X(j\omega))^2 \, d\omega = \int_{0}^{\infty} (2e^{-2t}e^{-t})^2 \, dt = \int_{0}^{\infty} (4e^{-4t} + e^{-3t}) \, dt
\]

\[= \left[-e^{-4t} - \frac{1}{2}e^{-2t} + \frac{4}{3}e^{-3t}\right]_0^\infty = 1 + \frac{1}{2} - \frac{4}{3} = \frac{1}{6}\]
**Part b.** Consider the causal and stable LTI system whose input $x(t)$ and output $y(t)$ are related by the linear constant-coefficient differential equation

$$\frac{d^2y(t)}{dt^2} + 2\frac{dy(t)}{dt} + y(t) = x(t).$$

Find the unit step response.

$$s(t) = -te^{-t}u(t) - e^{-t}u(t) + u(t)$$
Work page for Problem 3

\[ H(j\omega) = \frac{1}{(j\omega)^2 + 2(j\omega) + 1} = \frac{1}{(j\omega + 1)^2} \]

\[ \Rightarrow h(t) = te^{-t}u(t) \]

\[ S(t) = \int_{-\infty}^{t} h(\tau) d\tau = \int_{-\infty}^{t} \tau e^{-\tau} u(\tau) d\tau \]

Unit step response

\[ = \begin{cases} 
\int_{0}^{t} \tau e^{-\tau} d\tau & , \quad t \geq 0 \\
0 & , \quad t < 0
\end{cases} \]

\[ \int_{0}^{t} \tau e^{-\tau} d\tau = -\tau e^{-\tau} \bigg|_{0}^{t} - \int_{0}^{t} (-e^{-\tau}) d\tau \]

Integration by parts

\[ = (-\tau e^{-\tau} - e^{-\tau}) \bigg|_{0}^{t} = -te^{-t} - e^{-t} + 1 \]

Thus,

\[ S(t) = \begin{cases} 
-te^{-t} - e^{-t} + 1 & , \quad t \geq 0 \\
0 & , \quad t < 0
\end{cases} \]

Or equivalently,

\[ S(t) = -te^{-t}u(t) - e^{-t}u(t) + u(t) \]
PROBLEM 4 (18%)

Let $x[n]$ be a discrete-time signal whose Fourier transform between $-\pi$ and $\pi$ is

$$X(e^{j\omega}) = \begin{cases} \left(\frac{4}{\pi} \omega\right)^2, & |\omega| < \frac{\pi}{4} \\ 0, & \frac{\pi}{4} \leq |\omega| \leq \pi \end{cases}$$

\[\cdots \begin{array}{cccccccc} X(e^{j\omega}) \end{array} \cdots\]

$$-2\pi \quad -\pi \quad -\pi/4 \quad \pi/4 \quad \pi \quad 2\pi \quad \omega$$

**Part a.** Find $x[0]$ and $\sum_{n=-\infty}^{\infty} |x[n]|^2$.

$$x[0] = \frac{1}{2}$$

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{20}$$

From the definition of the inverse DTFT

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{2\pi} X(e^{j\omega}) e^{j\omega n} \, d\omega$$

$$x[0] = \frac{1}{2\pi} \int_{-\pi}^{2\pi} X(e^{j\omega}) \, d\omega$$

$$= \frac{1}{\pi} \int_{0}^{\pi/4} \left(\frac{4}{\pi} \omega\right)^2 \, d\omega = \frac{1}{\pi} \left(\frac{4}{\pi}\right)^2 \frac{1}{3} \left(\frac{\pi}{4}\right)^3 = \frac{1}{12}$$
Using Parseval's relation

\[
\sum_{n=-\infty}^{+\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega
\]

\[
= \frac{1}{\pi} \int_{0}^{\pi} \left( \frac{4}{\pi} \omega \right)^4 d\omega = \frac{1}{\pi} \left( \frac{4}{\pi} \right)^4 \frac{1}{5} \left( \frac{\pi}{4} \right)^5
\]

\[
= \frac{1}{20}
\]
Part b. Consider a second signal 

\[ y[n] = (-1)^n x[n]. \]

Find \( y[0] \) and \( \sum_{n=-\infty}^{+\infty} |y[n]|^2 \).

\( y[0] = \frac{1}{12} \)

\[ \sum_{n=-\infty}^{+\infty} |y[n]|^2 = \frac{1}{20} \]

Multiplying \( x[n] \) by \((-1)^n\) simply shifts \( X(e^{j\omega}) \) by \( \pi \) without changing any areas. Also,

\[ y[0] = (-1)^0 x[0] = x[0] \]

\[ |y[n]| = |(-1)^n x[n]| = |x[n]| \]

Part c. Consider a third signal

\[ z[n] = x[n] + y[n]. \]

Find \( z[0] \) and \( \sum_{n=-\infty}^{+\infty} |z[n]|^2 \).

\( z[0] = \frac{1}{6} \)

\[ \sum_{n=-\infty}^{+\infty} |z[n]|^2 = \frac{1}{10} \]

The new signal \( z[n] \) consists of the sum of \( x[n] \) and \( y[n] \), so \( Z(e^{j\omega}) \) has twice the area (note that there is no overlap between \( X(e^{j\omega}) \) and \( Y(e^{j\omega}) \)). Hence, we expect our answers here to be the sum of the answers in part a, and part b.

Also,

\[ z[0] = x[0] + y[0] \]
Part d. Consider a fourth signal

\[ w[n] := x[n] \ast z[n]. \]

Find \( w[0] \) and \( \sum_{n=-\infty}^{+\infty} |w[n]|^2 \).

\[ w[0] = \frac{1}{2\pi} \]

\[ \sum_{n=-\infty}^{+\infty} |w[n]|^2 = \frac{1}{3\pi} \]

\[ w[n] = x[n] \ast z[n] \]

\[ = x[n] \ast \left( x[n] + y[n] \right) \]

\[ W(e^{j\omega}) = \left[ X(e^{j\omega}) \right]^2 + \left( X(e^{j\omega}) \cdot Y(e^{j\omega}) \right) \]

\[ = 0 \]

since \( X(e^{j\omega}) \) and \( Y(e^{j\omega}) \) don't overlap.

So,

\[ w[0] = \frac{1}{2\pi} \int_{2\pi} W(e^{j\omega}) \, d\omega \]

\[ = \frac{1}{2\pi} \left( \int_{2\pi} \left( X(e^{j\omega}) \right)^2 \, d\omega \right) = \frac{1}{2\pi} \]

\[ \sum_{n=-\infty}^{+\infty} w[n] = \frac{1}{2\pi} \int_{2\pi} |W(e^{j\omega})|^2 \, d\omega \]

\[ = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega})^4 \, d\omega = \frac{1}{\pi} \int_{0}^{\pi/4} \left( \frac{4}{\pi} \omega \right)^8 \, d\omega \]

\[ = \frac{1}{\pi} \left( \frac{4}{\pi} \right)^8 \cdot \frac{1}{9} \cdot \left( \frac{\pi}{4} \right)^9 = \frac{1}{3\pi} \]
Work page for Problem 4
PROBLEM 5 (18%)

Consider the system below.

We have

\( x(t) = \frac{2\pi}{\omega_M} \left[ \sin \left( \frac{\omega_M t}{2\pi t} \right) \right]^2 \)

\( p(t) = \sum_{k=-\infty}^{+\infty} \delta(t - kT) \)

\( \frac{2\pi}{T} = \frac{3}{2} \omega_M \)

\( H(j\omega) = \begin{cases} T, & |\omega| < \frac{3}{4} \omega_M \\ 0, & \text{elsewhere} \end{cases} \)

**Part a.** Sketch the magnitude and phase of the Fourier transform of \( x(t) \) on the axes below.
Work page for Problem 5

\[ x(t) = \frac{2\pi}{\omega M} \frac{\sin \left( \frac{\omega M t}{2} \right)}{t} + \frac{\sin \left( \frac{\omega M t}{2} \right)}{\pi t} \]

\[ x(j\omega) = \frac{1}{2\pi} \frac{2\pi}{\omega M} \quad \rightarrow \quad \omega \]

\[ = \quad \frac{1}{-\omega M} \quad \rightarrow \quad \omega \]
Part b. Sketch the magnitude and phase of the Fourier transform of $x_p(t)$ on the axes below.

Part c. Sketch the magnitude and phase of the Fourier transform of $x_q(t)$ on the axes below.
Work page for Problem 5

\[ P(j\omega) \]

Note \( \frac{3\omega_m}{\pi} < 2\omega_m \)

Aliasing occurs!

\[ x_q(t) = t \, x_r(t) \quad \leftrightarrow \quad X_q(j\omega) = j \frac{d}{d\omega} X_r(j\omega) \]

\[ \frac{1}{j} X_q(j\omega) \begin{array}{c} \uparrow \\ \frac{3}{4\pi} \end{array} \]

\[ \frac{-1}{\omega_m} = - \frac{1}{\omega_m} = -\frac{3}{4\pi} \]

\[ \begin{align*}
4 & - \frac{3j}{4\pi} = -\frac{\pi}{2} \\
4 & \frac{3j}{4\pi} = \frac{\pi}{2}
\end{align*} \]
**Part d.** Sketch the magnitude and phase of the Fourier transform of $x_r(t)$ on the axes below.

![Diagram of $|X_r(j\omega)|$ and $\angle X_r(j\omega)$]

**Part e.** Sketch the magnitude and phase of the Fourier transform of $y(t)$ on the axes below.

![Diagram of $|Y(j\omega)|$ and $\angle Y(j\omega)$]
Work page for Problem 5

\[ e^{-\frac{j\pi t}{T}} \leftrightarrow s\left(\omega - \frac{\pi}{T}\right) \quad \text{frequency shift} \]

\[ \frac{\pi}{T} = \frac{3}{4} \omega_m \]
PROBLEM 6 (10%) 

Part a. Consider the following MATLAB code.

```matlab
x=[1 2 1 2];
X=fft(x);
y=X(1);
```

What is the resulting value of \( y \)?

\[
y = \underline{6}
\]

Part b. Consider the frequency response

\[
H(e^{j\omega}) = \frac{1}{1 - \frac{1}{2}e^{-j\omega}}.
\]

Write MATLAB code that plots the magnitude of \( H(e^{j\omega}) \) for frequencies \(-\pi \leq \omega \leq \pi\) at increments of \( \frac{2\pi}{1000} \). Your code should produce the plot shown on the next page. Write your final MATLAB code neatly in the box below.

```matlab
%% Example Solution
w=[-pi:pi/1000:pi];
H=1./(1-(1/2)*exp(-j*w));
plot(w, abs(H));

%% Note that \( w \) can be defined correctly in other ways, such
%% as 'w=linspace(-pi, pi, 1001);'

%% The use of other functions is also possible. The command
%% below will plot the frequency response.
frez([1], [1 -1/2], 1001, 'whole');

%% Note that this will not yield the plot given in the test,
%% but is a valid method of plotting a frequency response.
%% It will plot the magnitude in dB and include a phase plot.
%% It will plot from 0 to 2pi.
```