Directions: The exam consists of 5 problems on pages 2 to 19. Please make sure you have all the pages. Enter all your work and your answers directly in the spaces provided on the printed pages of this booklet. Please make sure your name is on all sheets. DO IT NOW! All sketches must be adequately labeled. Unless indicated otherwise, answers must be derived or explained, not just simply written down. This examination is closed book, but students may use one 8 1/2 \times 11 sheet of paper for reference. Calculators may not be used. Note that the problems are not in the order of difficulty. Solve the problems that you can first.

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Problem 1  (10 Points)

Can the output of the LTI system below be \( y(t) = \frac{\sin W_3 t}{\pi t} \)? If yes, give the relation between \( W_1, W_2 \) and \( W_3 \). If not, briefly explain why.

\[
x(t) = \frac{\sin W_1 t}{\pi t} \quad h(t) = \frac{\sin W_2 t}{\pi t} \quad y(t) = \frac{\sin W_3 t}{\pi t}
\]

YES. Relation between \( W_1, W_2 \) and \( W_3 \): \( W_3 = \text{sign}(W_1 W_2) \min(|W_1|, |W_2|) \)

NO. Brief explanation:
We know that \( Y(j\omega) = X(j\omega)H(j\omega) \).

The Fourier transform of \( \frac{\sin(W_1t)}{\pi t} \) is of square waveform with height \( \text{sign}(W_1) \) and frequency range \((-|W_1|, |W_1|)\). Thus, \( Y(j\omega) \) is also a square waveform with height \( \text{sign}(W_1W_2) \) and frequency range \((-|W_3|, |W_3|)\) where \( W_3 = \text{sign}(W_1W_2) \min(|W_1|, |W_2|) \).

Through inverse Fourier transform, we have \( y(t) = \frac{\sin(W_3t)}{\pi t} \) and \( W_3 = \text{sign}(W_1W_2) \min(|W_1|, |W_2|) \).

Please notice that you will receive full credit if your answer is

\[ W_3 = \min(W_1, W_2) \]
Problem 2 (24 Points)

The Fourier transform of the Gaussian waveform

\[ g_0(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2} \]

is a Gaussian in the frequency domain given by

\[ G_0(j\omega) = e^{-\omega^2/2} \]

(a) What is the area under the curve \( g_0(t) \)?

\[
\text{Area} = 1 \\
\text{Area} = \int_{-\infty}^{\infty} g_0(t) \, dt = G_0(j\omega) \bigg|_{\omega=0} = 1
\]

(b) What is the Fourier transform of

\[ g_{\sigma}(t) = \frac{1}{\sqrt{2\pi} \sigma} e^{-t^2/(2\sigma^2)} \]

\[
G_{\sigma}(j\omega) = \text{sign}(\sigma) e^{-\sigma^2 \omega^2/2}
\]

Use the time scaling rule

\[ g_{\sigma}(t) = \frac{1}{\sigma} g_0\left(\frac{t}{\sigma}\right) \quad \Rightarrow \quad G_{\sigma}(j\omega) = \frac{|\sigma|}{\sigma} G_0(j\sigma\omega) = \text{sign}(\sigma) e^{-\sigma^2 \omega^2/2} \]
(c) What is the Fourier transform of

\[ g_1(t) = \frac{1}{\sqrt{2\pi}} te^{-t^2/2} \]

\[ G_1(j\omega) = -j\omega e^{-\omega^2/2} \]

Use the rule for derivative w.r.t \( t \)

\[ g_1(t) = -d\theta_0(t)/dt \quad \Rightarrow \quad G_1(j\omega) = -j\omega \theta_0(j\omega) = -j\omega e^{-\omega^2/2} \]

or use the rule for multiplication by \( t \)

\[ g_1(t) = t\theta_0(t) \quad \Rightarrow \quad G_1(j\omega) = j \frac{d\theta_0(j\omega)}{d\omega} = -j\omega e^{-\omega^2/2} \]

(d) What is the Fourier transform of

\[ g_2(t) = \frac{1}{\sqrt{2\pi}} (2t^2 - 1)e^{-t^2/2} \]

\[ G_2(j\omega) = (1 - 2\omega^2)e^{-\omega^2/2} \]

\[ g_2(t) = -2d\theta_1(t)/dt + \theta_0(t) \quad \Rightarrow \quad G_2(j\omega) = -2j\omega \theta_1(j\omega) + \theta_0(j\omega) = (1 - 2\omega^2)e^{-\omega^2/2} \]
Problem 3 (20 Points)

Consider a discrete time (DT) system defined by the following difference equation:

\[ y[n] = x[n - 1] + 2 \cos(\Omega_0 n) x[n] + x[n + 1] \]

where \( x[n] \) is the input, \( y[n] \) is the output, and \( 0 < \Omega_0 < 2\pi \).

(a) Express the Fourier transform of the output \( Y(e^{j\Omega}) \) in terms of the Fourier transform of the input \( X(e^{j\Omega}) \).

\[
Y(e^{j\Omega}) = X(e^{j(\Omega-\Omega_0)}) + 2 \cos \Omega X(e^{j\Omega}) + X(e^{j(\Omega+\Omega_0)})
\]

(b) Assuming that \( x[n] \) is real and even, will \( Y(e^{j\Omega}) \) be real and even?

YES

Brief Explanation:

**Time Domain:** If \( x[n] \) is real and even then \( 2 \cos(\Omega_0 n) x[n] \) is also real and even. Moreover, \( z[n] = x[n - 1] + x[n + 1] \) is real and even as shown below:

\[
z[-n] = x[-n - 1] + x[-n + 1] = x[n + 1] + x[n - 1] = z[n]
\]

Therefore, \( y[n] \) is real and even.

**Frequency Domain:** If \( x[n] \) is real and even, then \( X(e^{j\Omega}) \) is also real and even. Thus, \( Y(e^{j\Omega}) \) is real. Now, check if \( Y(e^{j\Omega}) \) is even.

\[
Y(e^{-j\Omega}) = X(e^{j(-\Omega-\Omega_0)}) + 2 \cos(-\Omega) X(e^{-j\Omega}) + X(e^{j(-\Omega+\Omega_0)})
\]

\[
= X(e^{j(\Omega+\Omega_0)}) + 2 \cos \Omega X(e^{j\Omega}) + X(e^{j(\Omega-\Omega_0)})
\]

\[
= Y(e^{j\Omega})
\]
(c) For the input \( x[n] = e^{jn\pi} = (-1)^n \), the output \( y[n] \) of the system described by the difference equation above can easily be verified NOT to be zero, although the following three statements imply \( y[n] = 0 \). Identify which statements are incorrect and prove why. Define \( h[n] \) as the unit sample response of the system.

**Statement 1.** \( h[n] = \delta[n - 1] + 2\delta[n] + \delta[n + 1] \)

**Statement 2.** The Fourier transform of \( h[n] \) is \( H(e^{j\Omega}) = 2 (1 + \cos\Omega) \)

**Statement 3.** \( y[n] = e^{jn\pi} H(e^{j\Omega}) \)

**Incorrect Statements:** 3

**Proof (only for incorrect statements):** It is asked to prove only for incorrect statements. Therefore, proving that statement 3 is incorrect by proving that statements 1 and 2 are correct does not give partial credit. The main point here is to realize that the system defined by the difference equation is **not TI** due to the multiplication with \( \cos(\Omega_0 n) \). The complex exponential \( e^{j\Omega_0 n} \) is an eigenfunction for LTI systems. Therefore, if the system was LTI, statement 3 would have been correct. However, for non LTI systems, this relation does not hold in general. Statement 1 is correct and can be verified easily by substituting \( x[n] = \delta[n] \) in the difference equation and realize that \( 2\cos(\Omega_0 n)\delta[n] = 2\delta[n] \). Statement 2 is just the DTFT of \( h[n] \) from statement 1.
Problem 4 (30 Points)

Consider the following system

\[ x_c(t) \xrightarrow{\text{Impulse to Sample}} x_d[n] = x_c(nT) \xrightarrow{\text{Sample to Impulse}} y(t) \]

The Fourier Transform of \( x_c(t) \) is given by

\[ X_c(j\omega) = \frac{1}{T} \left[ \frac{\sin(\frac{\pi}{4} - \frac{\omega}{4})}{\sin(\frac{\pi}{4})} \right] \]

Assume that \( p(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT) \)

(a) Sketch \( X_d(e^{j\Omega}) \) for \( -\pi \leq \Omega \leq +\pi \) and determine the maximum value of \( \omega_a \) and minimum value of \( \omega_b \) such that \( y(t) = x_c(t) \).

maximum \( \omega_a = \frac{\pi}{4T} \)

minimum \( \omega_b = \frac{3\pi}{4T} \)
Work Page

Part (a)

Sol:

In general, $X_d(e^{j\Omega}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X_c(j\Omega - \frac{2\pi k}{T})$. Since $X_c(j\omega) = 0 \forall |\omega| > \pi/T$, there is no aliasing. Therefore, $X_d(e^{j\Omega}) = \frac{1}{T} X_c(j\Omega)$ for $|\Omega| < \pi$ as plotted. Please notice that $X_d(e^{j\Omega})$ is periodic $2\pi$ though we only plot it within $(-\pi, \pi)$.

Define $y_p(t)$ to be the output of the “Samples to Impulse.” We have $Y_p(j\omega) = X_d(e^{j\omega T})$ as plotted below (It is periodic $2\pi/T$)

To recover $X(j\omega)$ from $Y_p(j\omega)$, the bandpass filter should have height $T$ and $\omega_a \in [0, \frac{\pi}{4T}]$, $\omega_b \in [\frac{3\pi}{4T}; \frac{5\pi}{4T}]$

$max(\omega_a) = \frac{\pi}{4T}$, \hspace{1cm} $min(\omega_b) = \frac{3\pi}{4T}$
Instead of having $x(t)$ at the input, we now have $\hat{x}(t) = x(t) + r(t)$ where $r(t)$ is an arbitrary periodic interference signal with fundamental period $T_r$.

For all the remaining parts of this problem, assume $\omega_a = \frac{2\pi}{9T}$ and $\omega_b = \frac{8\pi}{9T}$.

(b) Find the output $\hat{y}(t)$ for the case $T_r = T$.

$\hat{y}(t) = x(t)$

Sol:

If $r(t)$ is periodic with $T_r = T$, then $r[n] = r(nT)$ is a constant, say, $a$. Therefore, $\hat{x}[n] = x_c(nT) + a$. Define $\hat{y}_p(t)$ to be the output of the “Samples to Impulse.” We have $\hat{Y}_p(j\omega) = \hat{X}_d(e^{j\omega T})$ as plotted below.

However, the DC component will be removed by the filter, therefore $\hat{y}(t) = x_c(t)$. 
(c) Assume that $T_r = N \cdot T$, where $N \geq 1$ is an integer. Specify all values of $N$, such that $\hat{y}(t) = x(t)$.

$$N = 1, 2$$

**Sol:**

Now $r(t)$ is periodic with $T_r = N \cdot T$, therefore sampling it every $T$ will obtain $r[n]$ which is a periodic signal with period $N$. Therefore,

$$\hat{X}_d(e^{j\Omega}) = X_d(e^{j\Omega}) + R_d(e^{j\Omega})$$

(2)

where $R_d(e^{j\Omega}) = \sum_k 2\pi r_k \delta(\Omega - k \frac{2\pi}{N})$ and $\{r_k\}_{k=0}^N$ are the Fourier series coefficients of $r[n]$. In order to have $\hat{y}(t) = x(t)$ we must make sure that none of the components of $R_d(e^{j\Omega})$ will be in the region $(-\frac{8\pi}{9T}, -\frac{2\pi}{9T})$ or $(-\frac{2\pi}{9T}, \frac{8\pi}{9T})$ (due to the passbnd region of the filter $(-\frac{8\pi}{9T}, -\frac{2\pi}{9T}), (\frac{2\pi}{9T}, \frac{8\pi}{9T})$) and this is only TRUE for $N = 1, 2$.

The case for $N = 1$ is plotted in part (b). The case for $N = 2$ is as below.
(d) Find a specific periodic signal \( r(t) \) with a fundamental period \( T_r = 10T \) such that \( \hat{y}(t) = x(t) \).
(Note that this does not prove/disprove that \( \hat{y}(t) = x(t) \) for any arbitrary periodic signal with fundamental period \( T_r = 10T \)).

\[
r(t) = \cos \left( \frac{2\pi}{10} t \right)
\]

**Sol:**

Part (c) was dealing with an arbitrary periodic signal. Here we can choose a specific periodic signal with \( T_r = 10T \) such that \( \hat{y}(t) = x(t) \). If we choose \( r(t) \) to be \( \cos(\frac{2\pi}{10} t) \), then

\[
r[n] = \cos(\frac{2\pi}{10} N),
\]

which has only 2 non zero Fourier Series coefficients, therefore,

\[
R_d(e^{j\Omega}) = \pi \delta \left( \Omega - \frac{2\pi}{10} \right) + \pi \delta \left( \Omega + \frac{2\pi}{10} \right) \quad |\Omega| < \pi
\]

These components are outside the region \( \left[ -\frac{8\pi}{9}, -\frac{2\pi}{9} \right] \) or \( \left[ \frac{2\pi}{9}, \frac{8\pi}{9} \right] \). Therefore \( \hat{y}(t) = x(t) \).
(e) Repeat part (c) where now $p(t)$ is as shown below.

\[
p(t) \quad \begin{array}{c}
\vdots \\
-3T \\
-2T \\
-T \\
0 \\
T \\
2T \\
3T \\
\vdots \\
\end{array}
\]

\[
N = 1, 2
\]

**Sol:**

\[
p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \quad \Rightarrow \quad \hat{x}_d[n] = \hat{x}_c(nT)
\]

\[
p'(t) = \sum_{n=-\infty}^{\infty} (-1)^n \delta(t - nT), \quad \Rightarrow \quad \hat{x}_d^{\text{new}}[n] = (-1)^n \hat{x}_c(nT) = (-1)^n \hat{x}_d[n]
\]

Thus,

\[
\hat{X}_d^{\text{new}}(e^{j\Omega}) = \hat{X}_d(e^{j(\Omega - \pi)})
\]

Since $X_d(e^{j\Omega}) = X_d(e^{j(\Omega - \pi)})$, $x(t)$ can be recovered from $y(t)$ through the BPF as long as we don’t have interference in the passband.

For $N = 1, 2$, the interference components are not in the passband. Therefore, in both cases, the noise components will be removed by the filter, thus $\hat{y}(t) = x_c(t)$. If $N \geq 3$, the spacing between two relative impulses is less than the passband width. Thus, noise cannot be removed.
Problem 5  (16 Points)

A discrete time signal $x[n]$ is given below (not drawn to scale)

where $a$, $b$, $c$, $d$, and $e$ are unknown real numbers. We are also told that

\[ \Re \{ X(e^{j\Omega}) \} = 2 + \cos(\Omega) + \cos(3\Omega) \]

and that

\[ \frac{1}{2\pi} \int_{-\pi}^{\pi} |\Im \{ X(e^{j\Omega}) \}|^2 d\Omega = 3 \]

Find $\Im \{ X(e^{j\Omega}) \}$. If there is more than one solution, find them all.

\[ \Im \{ X(e^{j\Omega}) \} = -\sin(\Omega) \pm 2\sin(2\Omega) + \sin(3\Omega) \]

Sol:

**Sol**

We know that $x_e[n] \xleftrightarrow{\cal F} \Re \{ X(e^{j\Omega}) \}$ and $x_o[n] \xleftrightarrow{\cal F} j\Im \{ X(e^{j\Omega}) \}$ while $x_e[n]$ and $x_o[n]$ are respectively the even and odd component of $x[n]$. 
\[ x[n] = a\delta[n + 3] + b\delta[n + 2] + c\delta[n] + d\delta[n - 1] + e\delta[n - 2] \]
\[ x_e[n] = \frac{x[n] + x[-n]}{2} \]
\[ = c\delta[n] + \frac{d}{2}(\delta[n - 1] + \delta[n + 1]) + \frac{(b + e)}{2}(\delta[n - 2] + \delta[n + 2]) + \frac{a}{2}(\delta[n - 3] + \delta[n + 3]) \]
\[ x_o[n] = \frac{x[n] - x[-n]}{2} \]
\[ = \frac{d}{2}(\delta[n - 1] - \delta[n + 1]) + \frac{(e - b)}{2}(\delta[n - 2] - \delta[n + 2]) - \frac{a}{2}(\delta[n - 3] - \delta[n + 3]) \]

Define \( X_e(e^{j\Omega}) = FT\{x_e[n]\} \) and \( X_o(e^{j\Omega}) = FT\{x_o[n]\} \) We have,
\[
X_e(e^{j\Omega}) = c + d\cos(\Omega) + 2(b + e)\cos(2\Omega) + a\cos(3\Omega) \\
X_o(e^{j\Omega}) = -jd\sin(\Omega) - j(e - b)\sin(2\Omega) + ja\sin(3\Omega) 
\]
Thus
\[
\Re\{X(e^{j\Omega})\} = X_e(e^{j\Omega}) = 2 + \cos(\Omega) + \cos(3\Omega) \\
\Rightarrow \quad c = 2, \quad d = 1, \quad b + e = 0, \quad a = 1 
\]

Since
\[
\frac{1}{2\pi} \int_{<2\pi>} |\Im\{X(e^{j\Omega})\}|^2 d\Omega = \frac{1}{2\pi} \int_{<2\pi>} |X_o(e^{j\Omega})|^2 d\Omega = \sum_{n=-\infty}^{\infty} x_0[n]^2 = 3
\]
we have
\[
\Im\{X(e^{j\Omega})\} = 2d^2 + 2\left(\frac{e - b}{2}\right)^2 + 2\left(\frac{a}{2}\right)^2 = 2\left(\frac{1}{2}\right)^2 + 2\left(\frac{e - b}{2}\right)^2 + 2\left(\frac{1}{2}\right)^2 = 3
\]
Thus, \( e - b = \pm 2 \) and
\[
X_o(e^{j\Omega}) = -j\sin(\Omega) \pm 2j\sin(2\Omega) + j\sin(3\Omega) \\
\Im\{X(e^{j\Omega})\} = X_o(e^{j\Omega})/j = -\sin(\Omega) \pm 2\sin(2\Omega) + \sin(3\Omega)
\]
We can also find out that
\[
a = 1 \quad b = -e = \pm 1 \quad c = 2 \quad d = 1
\]