Problem #1
Note that the duration of $y(t)$ is 4 units of time longer than that of $h(t)$. This fits with $T = 2$ for which the convolution of $x(t)$ and $h(t)$ will span the range from $-6$ to $6$. To find $A$ we need only determine $y(t)$ at $t = 0$. Thus, $y(0)$ equals the shaded area in the figure below multiplied by $A$.

The shaded area can be computed as the difference in areas of the large triangle and the two small triangles. Therefore, $A(0.5 \times 8 \times 1 - 2 \times 0.5 \times 2 \times 0.5) = 12$, so that $3A = 12$, and $A = 4$.

Problem #2

(a) First, we see that $z_1[n]$ is just $x_1[n]$ expanded along the time axis by a factor of 2, with zeros inserted at the odd samples. This same relationship holds for $z_2[n]$ and $x_2[n]$. Using the discrete-time Fourier transform property for time-scale expansion (see Table 5.1), we find that in the frequency domain,

$$Z_1(e^{j\omega}) = X_1(e^{j2\omega})$$
$$Z_2(e^{j\omega}) = X_2(e^{j2\omega}).$$

$Z_1(e^{j\omega})$ and $Z_2(e^{j\omega})$ are sketched below.
We can see from the problem that $Y(e^{j\omega})$ is a combination of the low-frequency spectrum of $Z_1(e^{j\omega})$ and the high-frequency spectrum of $Z_2(e^{j\omega})$. $Y(e^{j\omega})$ is sketched below.

(b) Since $w[n]$ is the result of passing $y[n]$ through a DT lowpass filter, $W(e^{j\omega})$ contains only the low frequency part of $Y(e^{j\omega})$.

To recover $X_1(e^{j\omega})$ from $W(e^{j\omega})$, we need to expand the width of the semi-circles by a factor of 2 in each period of the spectrum, in order for them to occupy the entire range of frequencies within one period. We can do so by downsampling $w[n]$ by a factor of 2. We also have to scale by 2 because of the $1/2$ factor which results from the downsampling process. The recovery procedure can therefore be summarized by the following equation,

$$x_1[n] = 2w[2n].$$
Problem #3

(a) We first find an expression for $z(t)$ in the time domain, in terms of $x_m(t)$ and $\cos(\omega_c t)$,

$$z(t) = 5y(t) + y^2(t) = 5(x_m(t) + \cos(\omega_c t)) + (x_m(t) + \cos(\omega_c t))^2 = 5x_m(t) + 5 \cos(\omega_c t) + 2x_m(t) \cos(\omega_c t) + \cos^2(\omega_c t).$$

$Z(j\omega)$ is just the sum of the Fourier transforms of each of the five terms in $z(t)$. The transform of each of the terms is listed below:

1. $5x_m(t) \longleftrightarrow 5X_m(j\omega)$
2. $5 \cos(\omega_c t) \longleftrightarrow 5\pi [\delta(\omega - \omega_c) + \delta(\omega + \omega_c)]$
3. $x_m^2(t) \longleftrightarrow \frac{1}{\pi} X_m(j\omega) * X_m(j\omega)$
4. $2x_m(t) \cos(\omega_c t) \longleftrightarrow X_m(j(\omega - \omega_c)) + X_m(j(\omega + \omega_c))$
5. $\cos^2(\omega_c t) \longleftrightarrow \pi\delta(\omega) + \frac{\pi}{2} [\delta(\omega - 2\omega_c) + \delta(\omega + 2\omega_c)]$

Since we are given that $X_m(j\omega)$ is a box from $\omega = -W$ to $\omega = W$ with a height of 1, the third item in the previous list (i.e. $\frac{1}{\pi} X_m(j\omega) * X_m(j\omega)$) will be a triangle which extends from $\omega = -2W$ to $\omega = 2W$, and has a height of $\frac{W}{\pi}$ at $\omega = 0$. We get $Z(j\omega)$ by combining all of these elements, as shown below.

(b) We now want to design a filter $H(j\omega)$ such that the output $w(t)$ has the form of $x_m(t)$ double-sideband amplitude-modulated (with carrier) on a carrier at $\omega_c$. This just means that we want $w(t) = x_m(t) \cos(\omega_c t) + A \cos(\omega_c t)$, where $A$ is a constant. Notice that the two parts of $w(t)$ are proportional to two elements of $z(t)$, namely $2x_m(t) \cos(\omega_c t)$ and $5 \cos(\omega_c t)$. Therefore, to get the correct answer, we need to introduce a scaling of $\frac{1}{2}$. In the frequency domain, $W(j\omega)$ then corresponds to the part of $Z(j\omega)$ which contains the modulated version of $X_m(j\omega)$ centered at $\omega_c$ and $-\omega_c$, scaled by $\frac{1}{2}$, with a pair of impulses at these two frequencies. Thus, we need $H(j\omega)$ to be the filter shown below.
Problem #4

There are a large number of ways to accomplish our desired goal; we will present one of the possible solutions. Divide $S_1$ into an upper and a lower branch. The upper branch will just transmit $X(j\omega)$ through the first communication channel. The channel will lowpass filter $X(j\omega)$, imposing a cutoff at frequency $W$. At $S_2$, this signal can be added in directly without any filtering or modulation.

Now, we must find a method for transmitting the remaining part of $X(j\omega)$. To do this, just modulate the frequencies we want to transmit into the available bandwidth. If we modulate $X(j\omega)$ with the carrier signal $2\cos(2Wt)$, the result will be two frequency-shifted copies of $X(j\omega)$,

$$
\mathcal{F}\{x(t) \times 2\cos(2Wt)\} = \frac{1}{2\pi} X(j\omega) * 2\pi (\delta(w - 2W) + \delta(w + 2W))
= X(j(\omega - 2W)) + X(j(\omega + 2W)).
$$

Transmit this modulated signal through the second communication channel. Again, the channel will act like a lowpass filter. In $S_2$, modulate the channel output with $2\cos(2Wt)$, bandpass filter the result to retain the frequency band of interest, and add it to the output of the upper channel. A block diagram of the proposed solution is shown on the following page.

Problem #5

We find the Laplace transform of both the input and the output,

$$
X(s) = \frac{1}{s+1} \quad \text{for } \Re(s) > -1,
$$

\[ Y(s) = K \left( \frac{1}{s+3} - \frac{1}{s-1} \right) \quad \text{for} \ -3 < \Re(s) < 1, \]

\[ Y(s) = \frac{-4K}{(s-1)(s+3)} \quad \text{for} \ -3 < \Re(s) < 1. \]

The ROC of the right-sided component of \( y(t) \) is \(-3 < \Re(s)\) and that of the left-sided part is \( \Re(s) < 1 \). Thus, the ROC of \( Y(s) \) is the overlap region \(-3 < \Re(s) < 1\). We find \( H(s) \) as follows

\[ H(s) = \frac{Y(s)}{X(s)} = \frac{-4K}{(s-1)(s+3)} = \frac{-4K(s+1)}{(s-1)(s+3)} \quad \text{for} \ -3 < \Re(s) < 1. \]

From the eigenfunction property of LTI systems, we know that \( x(t) = e^{s_0 t} \rightarrow y(t) = H(s_0)e^{s_0 t} \). Therefore, for \( x(t) = 1 = e^{0t} \), \( y(t) = H(0)e^{0t} = H(0) = \frac{8}{3} \). Since \( s = 0 \) is included in the ROC, the above statement is valid, and we can then use this fact to determine \( K \),

\[ \frac{8}{3} = H(0) = \frac{-4K(0+1)}{(0-1)(0+3)} = \frac{4K}{3}. \]

Thus, \( K = 2 \), giving

\[ H(s) = \frac{-8(s+1)}{(s-1)(s+3)} \quad \text{for} \ -3 < \Re(s) < 1. \]

Note that \( H(s) \) has a zero at \( s = -1 \) which also corresponds to the frequency of the exponential in \( x(t) \). Therefore, there is no component \( e^{-t} \) in \( y(t) \), because it has been “zeroed” out.

**Problem #6**

\( X(z) \) is a proper rational function in \( z^{-1} \). Therefore, we can expand it in a partial fraction expansion

\[ X(z) = \frac{1+3z^{-1}}{1+3z^{-1}+2z^{-2}} = \frac{1+3z^{-1}}{(1+z^{-1})(1+2z^{-1})} = \frac{1-3}{1-2} + \frac{1-3/2}{1-1/2}. \]

This gives

\[ X(z) = \frac{2}{1+z^{-1}} + \frac{-1}{1+2z^{-1}} \quad \text{for} \ 1 < |z| < 2. \]

Since the pole at \(-1\) is at the inside edge of the region of convergence (ROC) and the pole at \(-2\) is at the outside edge of the ROC,

\[ x[n] = 2(-1)^nu[n] + (-2)^nu[-n-1]. \]

**Problem #7**

(a) There are 4 zeros at \( z = 0 \). Poles are located at

\[ z = (a^4)^{1/4} = \pm \sqrt[4]{a^2} \]

which are the locations \{+a, −a, +ja, −ja\}. These are shown in the pole-zero diagram on the right.
(b) Since the unit-sample response is right-sided, the region of convergence must lie outside a circle that intersects the poles. Therefore, the region of convergence is for \( |z| > |a| \).

(c) Since \( h[n] \) is absolutely summable, the region of convergence must include the unit circle. Therefore, \( |a| < 1 \).

---

**Problem #8**

Since Bode plots only show positive values of \( \omega \), our discussion below concentrates only on the shape of the graphs for \( \omega > 0 \), with the knowledge that \( |H(j\omega)| \) is symmetric about \( \omega = 0 \).

(A) **Answer: 1**

There are two poles close to the \( j\omega \)-axis. Therefore, we expect to see a resonant peak in the Bode plot. Since \( H(s) \) has no zero at \( s = 0 \), the value of \( 20 \log_{10} |H(j\omega)| \) will be fairly constant for low frequencies, and will not change dramatically until \( \omega \) gets close to the pole. As \( \omega \) goes to infinity, the Bode plot should drop off at 40 dB/dec, since there are two more poles than zeros in the finite \( s \)-plane.

(B) **Answer: 4**

This pole-zero plot is almost identical to (A), except that a zero has been added at \( s = 0 \). Thus, there are some similarities between the Bode plots of (A) and (B), such as the resonant peak when \( \omega \) gets close to the pole location. However, since a zero has been added at \( s = 0 \), the magnitude plot will increase at 20 dB/dec for the low frequency portion of the Bode plot. In addition, as \( \omega \) goes to infinity, the Bode plot will decrease at 20 dB/dec since there is one more pole than zero in the finite \( s \)-plane.

(C) **Answer: 5**

In this case, the pole-zero plot indicates that the poles are on the real axis, and therefore, we can use the straight-line approximation to estimate the Bode plot. For this part, \( H(s) \) has the following form,

\[
H(s) = \frac{K}{(s+a)(s+b)},
\]

where \( b > a \). Thus, the Bode plot will be fairly constant for \( \omega < a \). For \( a < \omega < b \), the graph will decrease at 20 dB/dec, and for \( \omega > b \), the graph will become constant.

(D) **Answer: 2**

In this case, \( H(s) \) has the following form,

\[
H(s) = \frac{K(s+a)}{(s+b)},
\]

where \( b > a \). Thus, the Bode plot will be fairly constant for \( \omega < a \). For \( a < \omega < b \), the graph will increase at 20 dB/dec because of the zero, and for \( \omega > b \), the graph will become constant.
(E) **Answer: 6**  
In this case, $H(s)$ has the following form,

$$H(s) = \frac{K(s + a)}{s}.$$  

Because of the pole at $s = 0$, the Bode plot will decrease at 20 dB/dec for $\omega < a$. Then, for $\omega > a$, the graph will become constant because there are an equal number of poles and zeros in the finite $s$-plane.

---

**Problem #9**

(a) Since the input is a complex exponential for all time, we use the eigenfunction property of LTI systems. We first find the system function $H(s)$,

$$H(s) = 1 - \frac{2}{s + 2} = \frac{s}{s + 2} \quad \text{for } \Re(s) > -2.$$  

Since $s = -1$ is in the ROC of $H(s)$,  

$$y(t) = H(-1)e^{-t} = -e^{-t}.$$  

(b) The time domain functions $x(t)$ and $h(t)$ can be expressed as sinc functions and products of sinc functions. The Fourier transforms will then be rectangular boxes and convolutions of rectangular boxes. Consider the following transform pairs:

$$x(t) = \frac{\sin(4\pi t)}{4\pi t} \quad \longleftrightarrow \quad X(j\omega)$$

$$h_1(t) = \frac{\sin(2\pi t)}{2\pi t} \quad \longleftrightarrow \quad H_1(j\omega)$$

$$h_2(t) = \left(\frac{\sin(2\pi t)}{2\pi t}\right)^2 \quad \longleftrightarrow \quad H_2(j\omega)$$

$$h(t) = h_2(t - 2) \quad \longleftrightarrow \quad H(j\omega) = H_2(j\omega)e^{-2j\omega}$$
We know that \( Y(j\omega) = X(j\omega)H(j\omega) \). Notice that both \( X(j\omega) \) and \( H(j\omega) \) have non-zero values only over the range \(-4\pi < \omega < 4\pi\), and since \( X(j\omega) \) is constant over this range, it simply scales \( H(j\omega) \). Therefore,

\[
Y(j\omega) = \frac{1}{4} H(j\omega) = \frac{1}{4} H_2(j\omega) e^{-j2\omega}
\]

\( \implies y(t) = \frac{1}{4} \left( \sin 2\pi(t - 2) \right)^2. \)

(c) This problem can be solved using the \( z \)-transform. We first determine the \( z \)-transforms of \( x[n] \) and \( h[n] \),

\[
x[n] = \delta[n] + 2^n u[-n - 1] \quad \xRightarrow{z} \quad X(z) = 1 - \frac{1}{1 - 2z^{-1}} = \frac{-2z^{-1}}{1 - 2z^{-1}} \quad \text{for } |z| < 2
\]

\[
h[n] = 0.5^n u[n] \quad \xRightarrow{z} \quad H(z) = \frac{1}{1 - 0.5z^{-1}} \quad \text{for } |z| > 0.5.
\]

We could have also determined \( X(z) \) by using the time-reversal property, since \( x[n] = h[-n] \). Then,

\[
Y(z) = X(z)H(z) = \frac{-2z^{-1}}{(1 - 2z^{-1})(1 - 0.5z^{-1})} \quad \text{for } 0.5 < |z| < 2,
\]

and partial fraction expansion yields

\[
Y(z) = \frac{-\frac{3}{2}}{1 - 2z^{-1}} + \frac{\frac{3}{2}}{1 - 0.5z^{-1}} \quad \text{for } 0.5 < |z| < 2.
\]

By table lookup, the inverse \( z \)-transform is then given by

\[
y[n] = \frac{4}{3} \left[ 2^n u[-n - 1] + (1/2)^n u[n] \right] = \frac{4}{3} \left( \frac{1}{2} \right)^{|n|}.
\]

(d) The closed-loop transfer function is obtained with Black’s formula,

\[
H(s) = \frac{300}{s(s + 2)(s + 10)} = \frac{300}{s(s + 2)(s + 10) + 300} = \frac{300}{s^3 + 12s^2 + 20s + 300}.
\]

First, we check the stability of this system. All the coefficients are positive but \( 12 \times 20 < 300 \). Then, by the Routh-Hurwitz test, this system has poles in the right half of the \( s \)-plane, and hence, is unstable. Thus, \( \lim_{t \to \infty} y(t) = \infty \).

(e) This problem is most readily solved by evaluating the convolution integral. Since \( y(t) \) is only required at \( t = 1 \), it is helpful to sketch \( h(\tau) \) and \( x(1 - \tau) \) versus \( \tau \). The value of \( y(1) \) will then be the integral of the product of \( x(1 - \tau) \) and \( h(\tau) \).

A plot of \( h(\tau) \) and \( x(1 - \tau) \) versus \( \tau \) shows that the product of \( h(\tau) \) and \( x(1 - \tau) \) is non-zero only over the region \(-2 < \tau < -1\).
Hence, we evaluate the integral only over this interval,

\[ y(1) = \int_{-2}^{-1} x(1 - \tau)h(\tau) \, d\tau = \int_{-2}^{-1} e^{\tau + 1} \, d\tau = 1 - e^{-1} \approx 0.632. \]

**Problem #10**

Two relatively efficient ways can be used to solve this problem. One way is to find the difference equation and solve it iteratively for \( y[0] \) and \( y[1] \). Cross multiplication of the system function allows recognition of the difference equation as

\[ y[n] - 2y[n-1] - \frac{1}{4}y[n-2] + \frac{1}{2}y[n-3] = x[n] - x[n-1]. \]

Note, since \( x[n] = 0 \) for \( n < 0 \) and the system is causal, \( y[n] = 0 \) for \( n < 0 \). Therefore, from the difference equation and the input, \( y[0] = x[0] = 1 \). Therefore,

\[ y[1] - 2y[0] = x[1] - x[0], \]
\[ y[1] - 2 = \frac{1}{2} - 1, \]

from which \( y[1] = 3/2 \).

Another method is to find \( Y(z) \) and use synthetic division as follows. First,

\[ X(z) = \frac{1}{1 - 0.5z^{-1}}. \]

Therefore,

\[ Y(z) = \frac{1 - z^{-1}}{(1 - 0.5z^{-1})(1 - 2z^{-1} - \frac{1}{4}z^{-2} + \frac{1}{2}z^{-3})}, \]

\[ = \frac{1 - z^{-1}}{1 - \frac{5}{2}z^{-1} + \frac{3}{4}z^{-2} + \frac{5}{8}z^{-3} - \frac{1}{4}z^{-4}}. \]

Only the coefficients of \( z^0 \) and \( z^{-1} \) in the denominator are needed to find \( y[0] \) and \( y[1] \),

\[ 1 - \frac{5}{2}z^{-1} + \frac{3}{4}z^{-2} + \frac{5}{8}z^{-3} - \frac{1}{4}z^{-4} \cdot \frac{1 + \frac{3}{2}z^{-1} + \cdots}{\sqrt{1 - \frac{5}{2}z^{-1} + \cdots}} \cdot \frac{-(1 - \frac{5}{2}z^{-1} + \cdots)}{\frac{5}{2}z^{-1} + \cdots}. \]

Therefore, \( y[0] = 1 \) and \( y[1] = \frac{3}{2} \).
Problem #11

(a) From Black’s formula

\[ H(s) = \frac{K}{s-1} \frac{1}{1 + K(s+1)/(s-1)^2} = \frac{K}{s^2 - 2s + 1 + Ks + K} = \frac{K}{s^2 + (K-2)s + (K+1)}. \]

(b) A necessary and sufficient condition that the roots of a quadratic have negative real parts is that all the coefficients of the polynomial have the same sign. For \( K > 0 \), this condition is satisfied if \( K - 2 > 0 \) which implies that \( K > 2 \).

(c) From the denominator polynomial of \( H(s) \), the poles are located at

\[ s = -\frac{K-2}{2} \pm \frac{\sqrt{(K-2)^2 - 4(K+1)}}{2}. \]

From the form of the impulse response we see that the poles must be coincident, i.e., there is a second-order pole. This occurs when the discriminant is zero, i.e.,

\[ K^2 - 4K + 4 - 4K - 4 = 0 \implies K(K-8) = 0. \]

Thus, there are two values of \( K \) for which the poles are coincident, \( K = 0 \) and \( K = 8 \). At \( K = 0 \), the closed-loop poles equal the open-loop poles which occur at \( s = 1 \) (which is clear from the root-locus diagram). However, for \( K = 0 \), \( H(s) = 0 \) and \( h(t) = 0 \) so this solution is trivial. At \( K = 8 \), the poles are coincident at

\[ s = -\frac{8 - 2}{2} = -3 \]

which is also consistent with the root-locus diagram. For \( K = 8 \),

\[ H(s) = \frac{8}{(s+3)^2}, \]

and

\[ h(t) = 8te^{-3t}u(t) \]

so that the answers are \( K = 8 \), \( \alpha = 3 \), \( A = 8 \).

Problem #12

(a) Using Black’s formula, we get

\[ H(z) = \frac{\frac{1}{1+\frac{1}{2}z^{-1}}}{1 + K \frac{\frac{1}{1+z^{-1}}}{\frac{1}{1+\frac{1}{2}z^{-1}} - \frac{z^{-1}}{1-\frac{1}{2}z^{-1}}}}. \]
Simplifying the expression, gives

\[ H(z) = \frac{z^{-1} - \frac{1}{2}z^{-2}}{1 - \frac{1}{4}z^{-2} + Kz^{-2}} = \frac{z - \frac{1}{2}}{z^2 + (K - \frac{1}{4})}. \]

(b) To determine the locus of the poles of \( H(z) \), we need to find the roots of the denominator, \( i.e. \) the values of \( z \) for which \( z^2 + (K - \frac{1}{4}) = 0 \). Solving this equation (as a function of \( K \)), we get that the poles are

\[ z = \pm \sqrt{\frac{1}{4} - K}. \]

Now, to plot the root locus, we plot the values of \( z \) described by the above equation for different values of \( K \). In this case, we are asked to plot the root locus only for positive values of \( K \). For \( K = 0 \), the poles are located at \( z = \frac{1}{2} \) and \( z = -\frac{1}{2} \). As \( K \) increases to \( \frac{1}{4} \), the poles move toward the origin. For \( K > \frac{1}{4} \), the roots are purely imaginary and will increase in magnitude as \( K \) increases. The root locus is sketched below; the arrows indicate the direction of increasing \( K \).

Since the system is causal, it will be stable as long as the poles are inside the unit circle. This requirement is satisfied for

\[ \left| \pm \sqrt{\frac{1}{4} - K} \right| < 1. \]

Thus, for positive values of \( K \), the system is stable if

\[ 0 \leq K < \frac{5}{4}. \]