Analysis of Markov Chains

(Read: 7.2-7.4)

- statistical balance equations
- time to absorption
- probability of absorption
- classification of states
- steady-state distribution
- frequency of visits

Example: A Two-State Markov Chain

1. State 1
2. State 2
3. The absorbing state

Times until absorption: \( T_1, T_2 \)

Easy part: \( T_2 = 0 \)

Expected absorption times: \( \mu_i = E[T_i] \)

Balance: \( \mu_1 = 1 + (1 - p)\mu_1 + p\mu_2 \)

Answer: \( \mu_1 = 1/p \)

Example: time until “66” or “6*6”

\[
\begin{align*}
a &= 1 + \frac{5}{6}a + \frac{1}{6}b \\
b &= 1 + \frac{5}{6}c \\
c &= 1 + \frac{5}{6}a \end{align*}
\]

\( a = \frac{282}{11} \)

Example: Infinite Absorption Time

1. State 1
2. State 2
3. State 3
4. State 4
5. The absorbing state

Starting at “3”, time to absorption is infinite with probability ½!
Balance Equations for Absorption Times

- Markov Chain: $X_0, X_1, \ldots \in \{1, \ldots, m\}$
- $A$: a subset of $\{1, \ldots, m\}$
- Assume: $A$ accessible from everywhere
- $T_i^A$: first time to reach $A$ from state $i$
- $\mu_i = \mathbb{E}[T_i^A]$

$$
\mu_i = \begin{cases} 
0, & i \in A, \\
1 + \sum_j p_{ij} \mu_j, & i \notin A.
\end{cases}
$$

Example: Probability of Absorption

$$a_k = \mathbb{P}(\text{end in 1 starting from } k)$$

Balance Equations:

$$a_k = \frac{1}{2} a_{k-1} + \frac{1}{2} a_{k+1}, \quad (1 < k < m)$$

$a_1 = 1, \quad a_m = 0$

Solution:

$$a_k = \frac{m-k}{m-1}$$

Balance for Absorption: Derivation

- $p_{i,1}, \mu_1$
- $p_{i,2}, \mu_2$
- $p_{i,m}, \mu_{m-1}$
- $\mu_i$

Balance Equations For Absorption Probabilities

- Markov Chain: $X_0, X_1, \ldots \in \{1, \ldots, m\}$
- $A$: a subset of $\{1, \ldots, m\}$
- $C$: states from which $A$ is inaccessible
- $a_i = \mathbb{P}(\text{getting to } A \text{ from state } i)$

$$a_i = \begin{cases} 
0, & i \in C, \\
1, & i \in A, \\
\sum_j p_{ij} a_j, & \text{otherwise.}
\end{cases}$$
Example: Steady-State Distribution

\[ \pi_k = \lim_{t \to \infty} \mathbf{P}(X_t = k) \]

Balance Equations:

\[ \pi_k = 0.5\pi_{k-1} + 0.5\pi_{k+1} \quad (1 < k < m) \]
\[ \pi_1 = 0.5\pi_1 + 0.5\pi_2 \]
\[ \pi_m = 0.5\pi_m + 0.5\pi_{m-1} \]

Solution: \[ a_k = \frac{1}{m} \]

Example: No Steady-State Distribution

The probabilities of being at “1” or “2” do not converge, unless we start from “4”!

Example: “Mixed” Periodicity

If \( X_0 = 2 \) then

\[ \mathbf{P}(X_t = 2) = \begin{cases} 1, & t \text{ is even}, \\ 0, & t \text{ is odd}. \end{cases} \]

Recurrent States and Recurrent Classes

- State “i” is recurrent if it is accessible from every state which is accessible from “i”.
- The set of all states (necessarily recurrent) accessible from a given recurrent state is called a recurrent class.
- The state space of every Markov chain can be partitioned into recurrent classes and the set of transient (not recurrent) states.
Periodicity of a Recurrent Class

Period of a recurrent state is the greatest common divisor of all lengths of return trips originating in that state.

All states in the same recurrent class have the same period (called period of the class).

A recurrent state (or class) is called
- periodic when its period is greater than 1
- aperiodic when its period is equal to 1

Recognizing Periodicity

A recurrent class is periodic with period $d > 1$ if and only if it can be partitioned into $d$ groups $G_1, \ldots, G_d$ such that all single-step transitions from $G_1$ to $G_2$, from $G_2$ to $G_3$, etc. (the list ends with “$G_d$ to $G_1$”).

Existence of a Steady-State Limit

The limit $\pi_k = \lim_{t \to \infty} P(X_t = k)$ exists for all distributions of $X_0$ if and only if all recurrent states are aperiodic.

In general, the limits (and their existence) depend on the initial PMF $M_{X_0}(\cdot)$.

Steady-State PMF For a Recurrent Class

Assuming that all recurrent states are aperiodic, the steady-state probabilities

$$\pi_k = \lim_{t \to \infty} P(X_t = k)$$

for the elements “$k$” of a recurrent class $A$ are uniquely defined by the equations

$$\pi_k = \sum_{i \in A} \pi_i P_{ik} \quad (k \in A)$$

$$\sum_{i \in A} \pi_i = P(\text{reaching } A)$$
Balance for Distribution: Derivation

\[ \pi_1 \quad 1 \quad \pi_2 \quad 2 \quad \pi_{m-1} \quad m-1 \quad \pi_m \quad m \]
\[ p_{1,k} \quad p_{m,k} \quad \pi_k \]

\[ m \]

Example: two recurrent classes

\[ \begin{array}{c}
1 \\
2 \\
3 \\
4 \\
5 \\
\end{array} \]

\[ \begin{array}{c}
\text{recurrent class } \{1,2\} \\
\text{transient state} \\
\text{recurrent class } \{4,5\} \\
\end{array} \]

Balance Equations:
\[ \pi_1 = \pi_2, \quad \pi_4 = \pi_5, \quad \pi_3 = 0 \]

If \( P(X_0 = 2) = P(X_0 = 3) = 1/2 \):
\[ \pi_1 = \pi_2 = \frac{3}{8}, \quad \pi_4 = \pi_5 = \frac{1}{8}, \quad \pi_3 = 0 \]

Balance of Expected Visit Frequencies

\[ \Lambda_k(T) = \{ t \in \{1, \ldots, T \} : X_t = k \} \]
\[ N_k(T') = \text{no. of elements in } \Lambda_k(T') \]
\[ \pi_k = \lim_{T \to \infty} \frac{1}{T} \mathbb{E}[N_k(T)] \]

Balance Equations:
\[ \pi_k = \sum_{i \in A} \pi_i P_{ik} \quad (k \in A) \]
\[ \sum_{i \in A} \pi_i = P(\text{reaching } A) \]

Example: Visit Frequencies With Periodicity

\[ 1 \quad 2 \quad 1/2 \quad 3 \quad 1/2 \quad 4 \quad 1 \]

Balance Equations:
\[ \pi_1 = \pi_2, \quad \pi_3 = 0 \]

If \( P(X_0 = 2) = P(X_0 = 3) = 1/2 \):
\[ \pi_1 = \pi_2 = \frac{3}{8}, \quad \pi_3 = 0, \quad \pi_4 = 1/4 \]