LECTURE 6

• Readings: Sections 2.4-2.6

Lecture outline
• Review PMF, expectation, variance
• Conditional PMF
• Geometric PMF
• Total expectation theorem
• Joint PMF of two random variables
• Independence

Review
• Random variable $X$: function from sample space to the real numbers
• PMF (for discrete random variables): $p_X(x) = P(X = x)$, $\sum_x P(X = x) = 1$
• Expectation: $E[X] = \sum_x x \cdot p_X(x)$
• Geometric PMF
• Total expectation theorem
• Joint PMF of two random variables
• Independence

Conditional Expectation
• Definition:
$$p_{X|A}(x) = P(X = x|A) = \frac{P((X=x) \cap A)}{P(A)}$$
• Definition:
$$E[X|A] = \sum_x x \cdot p_{X|A}(x)$$

Conditional Expectation
• Recall: $p_{X|A}(x) = P(X = x|A)$
• Definition:
$$E[X|A] = \sum_x x \cdot p_{X|A}(x)$$

E[$X|X \geq 2] = \frac{2+3+4}{3}$
Geometric PMF
- $X$: Waiting time for the #1 bus at the MIT stop
  \[ p_X(k) = (1 - p)^{k-1} p, \quad k = 1, 2, \ldots \]
- What is the expected waiting time, $E[X]$?
- What is the expected waiting time conditioned on the fact that you have already waited 2 minutes?

Geometric PMF
- Expected time:
  \[ E[X] = \sum_{k=1}^{\infty} k \cdot p_X(k) = \sum_{k=1}^{\infty} k \cdot (1 - p)^{k-1} p \]
- Memoryless property:
  Given that $X > 2$, the r.v. $X - 2$ has same geometric PMF.
  \[ E[X|X > 2] = 2 + E[X] \]

Total Expectation Theorem
- Partition of sample space into disjoint events $A_1, A_2, \ldots, A_n$
  \[ P(B) = P(A_1)P(B|A_1) + \cdots + P(A_n)P(B|A_n) \]
  \[ E[X] = P(A_1)E[X|A_1] + \cdots + P(A_n)E[X|A_n] \]

Geometric R.V.
- Geometric example:
  \[ A_1 : \{X = 1\}, \quad A_2 : \{X > 1\} \]
  \[ E[X] = P(X = 1)E[X|X = 1] + P(X > 1)E[X|X > 1] \]
- Solve to get $E[X] = 1/p$
Joint PMFs
- $p_{X,Y}(x,y) = P(X = x \text{ and } Y = y)$
- $p_X(x) = \sum_y p_{X,Y}(x,y)$
- $p_{X|Y}(x|y) = P(X = x|Y = y) = \frac{p_{X,Y}(x,y)}{p_Y(y)}$
- $\sum_x \sum_y p_{X,Y}(x,y) = 1$
- $\sum_x p_{X|Y}(x|y) = 1$

Independent Random Variables
$p_{X,Y,Z}(x,y,z) = p_X(x)p_{Y|X}(y|x)p_{Z|X,Y}(z|x,y)$
- Random variables $X$, $Y$ and $Z$ are independent if (for all $x$, $y$ and $z$):
  $p_{X,Y,Z}(x,y,z) = p_X(x) \cdot p_Y(y) \cdot p_Z(z)$
- Example:
  Independent?
- What if we condition on $X \leq 2$ and $Y \geq 3$?