Problem 1: True or False?

1. If $L_1$ and $L_2$ are regular, then $L_1 \cup L_2$ is regular. **True**

2. If $L_1$ and $L_2$ are non-regular, then $L_1 \cap L_2$ is non-regular.
   **False.** Consider $L_1 = \{0^n1^n \mid n \geq 0\}$ and $L_2 = \{0^{n+1}1^n \mid n \geq 0\}$.

3. If $L_1$ is regular and $L_2$ is non-regular, then $L_1 \cup L_2$ is non-regular.
   **False.** $L_1 = \Sigma^*$ and $L_2$ any non-regular language.

4. If $L_1$ is regular, $L_2$ is non-regular, and $L_1 \cap L_2$ is regular, then $L_1 \cup L_2$ is non-regular.
   **True.** Write $L_2' = \{(L_1 \cup L_2) - L_1\} \cup (L_1 \cap L_2)$.

5. The following language is regular: The set of strings in $\{0, 1\}^*$ having the property that the number of 0’s and the number of 1’s differ by no more than 2.
   **False.**

6. The following language is regular: The set of strings in $\{0, 1\}^*$ having the property that in every prefix, the number of 0’s and the number of 1’s differ by no more than 2.
   **True.** A simple 5-state DFA accepts this language.

Problem 2: Proving non-regularity: the Pumping Lemma. Prove that the following languages are not regular.

1. $L_4 = \{0^i1^j2^k \mid i, j, k \geq 0 \text{ and if } i = 1 \text{ then } j = k\}$.
   **Solution:** Define $L_4' = \{1^j2^k \mid j, k \geq 0\} \cup \{0^i1^j2^k \mid i > 1, j, k \geq 0\}$. It is easily seen that $L_4'$ is regular. Now, observe that $L_4 - L_4' = \{01^j2^k \mid j \geq 0\}$ is not regular.

Problem 3: The size of the minimal DFA for a regular language $L$. Consider the regular language $L = \{w \mid w \text{ contains at least three } 1's\}$. Prove that any DFA for this language has at least 4 states.

**Solution:** The crucial fact to use is that, if strings $x$ and $y$ lead from the start state to the same state $q$, then for every string $z$, $xz \in L$ if and only if $yz \in L$. More formally, $\delta^*(q_0, x) = \delta^*(q_0, y)$ implies $\forall z \in \Sigma^*, xz \in L$ if and only if $yz \in L$. (Think about it and convince yourself that this is true).

Now, note that strings $\epsilon, 1, 11, 111$ must lead to different states. For instance, suppose $\delta(q_0, \epsilon) = \delta(q_0, 1)$. Then, setting $z = 11$, we see that $11 \notin L$ whereas $111 \in L$. This is a contradiction, and therefore the strings $\epsilon$ and 1 lead to different states.