6.342 Lecture 3 — February 17, 2009

Today:
• More on bases, matrix-vector representations and projection
• Sequences and discrete-time signals

Recall: Homework #1 is due tomorrow at beginning of lecture

Project assignment posted
• Several deliverables: Proposal, talk, report
• Discussions with staff are encouraged, and one meeting is mandatory

Recall: Inner product linear in first arg, conjugate-linear in second arg

Readings:
• Chapters 2-3 of The World of Fourier and Wavelets
  – Chapter 2 already posted, with many obvious flaws
  – Chapter 3 will be covered only very lightly in lecture

Matrix representations of vectors

\[ x = \sum_{k \in \mathbb{Z}} \alpha_k \varphi_k \quad \quad \quad y = \sum_{k \in \mathbb{Z}} \beta_k \varphi_k \]

\[ \langle x, y \rangle = \langle \sum_{k \in \mathbb{Z}} \alpha_k \varphi_k, \sum_{i \in \mathbb{Z}} \beta_k \varphi_i \rangle = \sum_{k \in \mathbb{Z}} \sum_{i \in \mathbb{Z}} \alpha_k \beta_i^* \langle \varphi_k, \varphi_i \rangle = \beta^* G \alpha \]

\[ G = \begin{bmatrix}
  \langle \varphi_{-1}, \varphi_{-1} \rangle & \langle \varphi_{0}, \varphi_{1} \rangle & \langle \varphi_{1}, \varphi_{-1} \rangle & \cdots \\
  \langle \varphi_{-1}, \varphi_{0} \rangle & \langle \varphi_{0}, \varphi_{0} \rangle & \langle \varphi_{1}, \varphi_{0} \rangle & \cdots \\
  \langle \varphi_{-1}, \varphi_{1} \rangle & \langle \varphi_{0}, \varphi_{1} \rangle & \langle \varphi_{1}, \varphi_{1} \rangle & \cdots \\
  \cdots & \cdots & \cdots & \ddots
\end{bmatrix} \]
Matrix representations of linear operators

\[
\beta_k = \left\langle A \left( \sum_{i \in \mathbb{Z}} \alpha_i \varphi_i \right), \tilde{\psi}_k \right\rangle \equiv \left\langle \left( \sum_{i \in \mathbb{Z}} \alpha_i A \varphi_i \right), \tilde{\psi}_k \right\rangle \equiv \sum_{i \in \mathbb{Z}} \alpha_i \left\langle A \varphi_i, \tilde{\psi}_k \right\rangle
\]

\[
\begin{bmatrix}
\vdots \\
\beta_{-1} \\
\beta_0 \\
\beta_1 \\
\vdots 
\end{bmatrix} =
\begin{bmatrix}
\vdots \\
\left\langle A \varphi_{-1}, \tilde{\psi}_{-1} \right\rangle \\
\left\langle A \varphi_0, \tilde{\psi}_0 \right\rangle \\
\left\langle A \varphi_1, \tilde{\psi}_1 \right\rangle \\
\vdots 
\end{bmatrix}
\begin{bmatrix}
\vdots \\
\alpha_{-1} \\
\alpha_0 \\
\alpha_1 \\
\vdots 
\end{bmatrix}
\]

---

**Last time**

**Definition 1.19 (Projection operator).** An operator \( P \) is called idempotent if \( P^2 = P \). A linear operator that is idempotent is a projection operator.

**Definition 1.20 (Orthogonal projection operator).** A projection operator that is self-adjoint is an orthogonal projection operator.

**Theorem 1.1 (Projection theorem [42]).** Let \( W \) be a closed subspace of the Hilbert space \( H \). Corresponding to any vector \( x \in H \), there exists a unique vector \( w_0 \in W \) such that \( \| x - w_0 \| \leq \| x - w \| \) for all \( w \in W \). A necessary and sufficient condition for \( w_0 \) to be the unique minimizing vector is that \( x - w_0 \perp W \). The operator \( P_W \) mapping \( x \) into \( w_0 \) is an orthogonal projection operator.
Projection using bases

Let \( \{\varphi_k\}_{k \in I} \) be a basis for subspace \( V \) of Hilbert space \( H \). How can we represent the projection onto \( V \)?

Biorthogonal expansion/reconstruction

Let \( H \) be a Hilbert space.
Let \( \{\varphi_k\} \subset H \) be a linearly independent set.
Let \( \{\tilde{\varphi}_k\} \subset H \) be a set biorthogonal to \( \{\varphi_k\} \):

\[
\langle \tilde{\varphi}_i, \varphi_j \rangle = \delta_{i-j}
\]

Then

- \( \{\tilde{\varphi}_k\} \) is a linearly independent set
- \( x = \sum_k \langle x, \tilde{\varphi}_k \rangle \varphi_k \) for all \( x \in \text{span}(\{\varphi_k\}) \) with uniqueness of coefficients
- \( x = \sum_k \langle x, \varphi_k \rangle \tilde{\varphi}_k \) for all \( x \in \text{span}(\{\tilde{\varphi}_k\}) \) with uniqueness of coefficients
Biorthogonal expansion/reconstruction

Let \( \{ \varphi_k \} \subset H \) be linearly independent, \( \{ \tilde{\varphi}_k \} \) and \( \{ \varphi_k \} \) biorthogonal, \( V = \text{span}(\{\varphi_k\}) \).

\( x \in H \) can be written uniquely as

\[
x = x_{V\perp} + \sum_k \langle x, \tilde{\varphi}_k \rangle \varphi_k, \quad \langle x_{V\perp}, \varphi_i \rangle = 0 \quad \forall \ i
\]

Having \( \tilde{\varphi}_k \in V \) for all \( k \) is special: Then

\( \hat{x} = \sum_k \langle x, \tilde{\varphi}_k \rangle \varphi_k \) is the orthogonal projection of \( x \) onto \( V \), i.e., \( \|x - \hat{x}\| \) is minimum.

Discrete time

We now work (almost) entirely in \( \ell^2(\mathbb{Z}) \):

- finite-energy sequences on \( \mathbb{Z} \)
- \( \mathbb{Z} \) is ordered!

Linear operators \( \ell^2(\mathbb{Z}) \rightarrow \ell^2(\mathbb{Z}) \) called "discrete-time systems"

\[
x_n \rightarrow \begin{bmatrix} A \end{bmatrix} \rightarrow y_n
\]

Matrix representations of operators

Time invariance
Basic discrete-time signals

discrete impulse $\delta_n = \begin{cases} 1, & \text{for } n = 0; \\ 0, & \text{otherwise} \end{cases}$

previously called $e_0$; shift to get $e_k$
(Kronecker? Dirac?)

unit step (or Heaviside) $u_n = \begin{cases} 1, & \text{for } n \geq 0; \\ 0, & \text{otherwise} \end{cases}$

(discrete) sinc $\text{sinc}_{T,n} = \begin{cases} \frac{\sin(n\pi/T)}{n\pi/T}, & \text{for } n \neq 0; \\ 1, & \text{for } n = 0 \end{cases}$

Deterministic correlation

**Autocorrelation** Given a sequence $x$, its autocorrelation $a^{11}$ is given by

$$a_n = \sum_{k \in \mathbb{Z}} x_k x_{k+n} = \langle x_k, x_{k+n} \rangle_k.$$  \hspace{1cm} (2.16)

The autocorrelation measures the similarity of a sequence with respect to shifts of itself. Note that $a_0 = \sum_k |x_k|^2 = \|x\|_2^2$, and $a_{-n} = a_{n}$.

**Crosscorrelation** Given two sequences $x$ and $y$, their crosscorrelation $c$ is given by

$$c_n = \sum_{k \in \mathbb{Z}} x_k y_{k+n} = \langle x_k, y_{k+n} \rangle_k.$$  \hspace{1cm} (2.17)

\hspace{1cm} \text{\textsuperscript{11}}This is the deterministic autocorrelation of a sequence. The stochastic case will be seen later.
Basic systems

**Shift** The shift-by-\( k \) operator is defined as:

\[
y_n = T x = x_{n-k},
\]

\[
[\ldots \ x_{-1} \ x_0 \ x_1 \ \ldots] \rightarrow [\ldots \ x_{k-1} \ x_k \ x_{k+1} \ \ldots].
\]

which simply delays \( x \) by \( k \) samples. While this is one of the simplest discrete-time systems, it is also the most important, as the whole concept of time processing is based on this simple operator. The shift-by-one is usually termed a *delay*.

**Accumulator** The output of the accumulator is the “integral” of the input:

\[
y_n = T x = \sum_{k=-\infty}^{n} x_k.
\]

Basic system properties

**Definition 2.2 (memoryless system).** A system \( y = T_n x \) is memoryless, if its output \( y_n \) at time \( n \) depends only on the input \( x_n \) at time \( n \):

\[
y_n = T x_n.
\]

**Definition 2.3 (shift-invariant system).** Given an input \( x_n \) and an output \( y_n \), a system is shift invariant if a shifted input produces a shifted output:

\[
y_n = T x_n \quad \Rightarrow \quad y_{n-k} = T x_{n-k}.
\]

**Definition 2.4 (causal system).** A causal system is a system whose output at time \( n \) depends only on the present and past inputs:

\[
y_n = T_n x_k, \quad k \leq n.
\]

**Definition 2.5 (BIBO stability).** A system is BIBO stable if, for every bounded input \( x \in \ell^\infty(\mathbb{Z}) \), the output \( y = T_n x \) is bounded as well:

\[
x \in \ell^\infty(\mathbb{Z}) \quad \Rightarrow \quad y \in \ell^\infty(\mathbb{Z}).
\]
Impulse response and convolution

Discrete-time Fourier transform

**Definition 2.6 (Discrete-time Fourier transform).** Given an infinite-length sequence \( x_n \), its discrete-time Fourier transform is given by

\[
X(e^{j\omega}) = \langle x, v_{\omega} \rangle = \sum_{n \in \mathbb{Z}} x_n e^{-j\omega n}, \quad \omega \in \mathbb{R}.
\]  

The inverse discrete-time Fourier transform is given by

\[
x_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \langle x, v_{\omega} \rangle v_{\omega} d\omega = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega, \quad n \in \mathbb{Z}.
\]  

To denote such a discrete-time Fourier transform pair, we write:

\[
x_n \xrightarrow{\text{DFT}} X(e^{j\omega}).
\]

The discrete-time Fourier transform \( X(e^{j\omega}) \) is a \( 2\pi \)-periodic function called the *spectrum* of the sequence \( x \).
### DTFT properties Table

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### Symmetries

- Conjugate \( x \):
  \( x^* \):
  \( X^*(e⁻ʲω) \)
- Conjugate, time-reversed \( x \):
  \( x^*₋ₙ \):
  \( X^*(e⁻ʲω) \)
- Real part of \( x \):
  \( R(xₙ) \):
  \( (X(e⁻ʲω) + X^*(e⁻ʲω))/2 \)
- Imaginary part of \( x \):
  \( jΩ(xₙ) \):
  \( (X(e⁻ʲω) - X^*(e⁻ʲω))/2 \)
- Conjugate-symmetric part of \( x \):
  \( (xₙ + x^*₋ₙ)/2 \):
  \( R(X(e⁻ʲω)) \)
- Conjugate-antisymmetric part of \( x \):
  \( (xₙ - x^*₋ₙ)/2 \):
  \( jΩ(X(e⁻ʲω)) \)

### Symmetries for real \( x \)

- \( X \) is conjugate symmetric:
  \( X(e⁻ʲω) = X^*(e⁻ʲω) \)
- Real part of \( X \) is even:
  \( R(X(e⁻ʲω)) = R(X(e⁻ʲ3ω)) \)
- Imaginary part of \( X \) is odd:
  \( Ω(X(e⁻ʲω)) = -Ω(X(e⁻ʲ3ω)) \)
- Magnitude of \( X \) is even:
  \( |X(e⁻ʲω)| = |X(e⁻ʲ3ω)| \)
- Phase of \( X \) is odd:
  \( \arg(X(e⁻ʲω)) = -\arg(X(e⁻ʲ3ω)) \)