Systems applied to 2-d signals

Signals: $\ell^2(\mathbb{Z}^2)$ is the set of finite-energy 2-d signals:

$$\ell^2(\mathbb{Z}^2) = \{ x : \sum_{n_2=-\infty}^{\infty} \sum_{n_1=-\infty}^{\infty} |x_{n_1,n_2}|^2 < \infty \}$$

A linear shift-invariant system $\ell^2(\mathbb{Z}^2) \to \ell^2(\mathbb{Z}^2)$ is characterized by its impulse response:

$$x_{n_1,n_2} \rightarrow [h_{n_1,n_2}] \rightarrow y_{n_1,n_2}$$

$$y_{n_1,n_2} = \sum_{\ell=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} h_{n_1-k,n_2-\ell} x_{k,\ell}$$

Counterpart to $z$ transform:

$$H(z_1, z_2) = \sum_{n_2=-\infty}^{\infty} \sum_{n_1=-\infty}^{\infty} h_{n_1,n_2} z_1^{-n_1} z_2^{-n_2}$$
Example

Simple lowpass filter:

\[
\begin{pmatrix}
\frac{1}{8} \\
\frac{1}{2} \\
\frac{1}{8}
\end{pmatrix}
\]

\[z \text{ domain: } H(z_1, z_2) = \frac{1}{2} + \frac{1}{8}(z_1 + z_1^{-1} + z_2 + z_2^{-2})\]

\[\text{Fourier domain:}
H(e^{j\omega_1}, e^{j\omega_2})
= \frac{1}{2} + \frac{1}{4}(\cos \omega_1 + \cos \omega_2)\]

Example

“Vertical” highpass filter:

\[
\begin{pmatrix}
1 \\
0 \\
-1
\end{pmatrix}
\]

\[z \text{ domain: } H(z_1, z_2) = z_2^{-1} - z_2\]

\[\text{Fourier domain:}
H(e^{j\omega_1}, e^{j\omega_2})
= -2j \sin \omega_2\]
Directionality and separability

Several highpass filters

(each has a zero at \((z_1, z_2) = (1, 1)\)):

\[
\begin{array}{cccccccc}
0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & \[0\] & 0 & 0 & \[0\] & 0 & -1 & \[0\] \\
0 & -1 & 0 & -1 & -1 & 0 & -1 & -1 & 1 & 0 & -1
\end{array}
\]

\[
\begin{array}{ccccccc}
0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & -1 & 0 & 1 \\
0 & \[0\] & 0 & 0 & \[0\] & 0 & -1 & \[0\] & 1 & 0 & \[0\] & 0 \\
0 & -1 & 0 & -1 & -1 & 0 & -1 & -1 & 0 & 1 & 0 & -1
\end{array}
\]

- directional, nonseparable
- “mixed” directionality separable

\[
\begin{array}{ccc}
-1 & 0 & 1 \\
0 & \[0\] & 0 \\
1 & 0 & -1
\end{array} = \begin{array}{c}
\[0\] \\
\[0\] \\
1
\end{array} \ast \begin{array}{c}
1 \\
0 \\
-1
\end{array}
\]
Separability and implementation

$h_{n_1,n_2}$ is separable when $h_{n_1,n_2} = h_{n_1}^{(1)} h_{n_2}^{(2)}$

\[
y_{n_1,n_2} = \sum_{\ell=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} h_{n_1-k,n_2-\ell} x_{k,\ell}
\]

\[
= \sum_{\ell=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} h_{n_1-k}^{(1)} h_{n_2-\ell}^{(2)} x_{k,\ell}
\]

\[
= \sum_{\ell=-\infty}^{\infty} h_{n_2-\ell}^{(2)} \sum_{k=-\infty}^{\infty} h_{n_1-k}^{(1)} x_{k,\ell}
\]

filter “in the first dimension”

filter “in the second dimension”

Implementing a separable LSI system (alt. view)

$h_{n_1,n_2}$ is separable when $h_{n_1,n_2} = h_{n_1}^{(1)} h_{n_2}^{(2)}$

Let $A_i$ be the operator for $h_{n_i}^{(i)}$, from column vectors to column vectors

\[
y_{n_1,n_2} = \sum_{\ell=-\infty}^{\infty} h_{n_2-\ell}^{(2)} \sum_{k=-\infty}^{\infty} h_{n_1-k}^{(1)} x_{k,\ell}
\]

column index fixed; filtering columns of x one at a time $A_1 x_{:,\ell}$

row index fixed; filtering rows of previous result one at a time $A_2^T$

\[
(A_2(A_1 x)^T)^T = ((A_1 x)^T)^T A_2^T = A_1 x A_2^T
\]
Two-channel analysis for 2-d signal

Can have a complementary pair of filters and throw away half of the samples from each branch and reconstruct perfectly. There exist orthogonal solutions, too.

Nonseparable two-channel analysis example

common separable downsampling by 4
Biorthogonal two-channel filter banks

With $L$ even, start with a length-$L$ FIR filter $g$ satisfying
\[ G(z)G(z^{-1}) + G(-z)G(-z^{-1}) = 2. \]

Then the following filters “work”:

- $h_n = (-1)^n g_{L-n}$
- $g_n = g_{-n}$
- $\tilde{h}_n = h_{-n}$
- $H(z) = -z^{-L+1}G(-z^{-1})$
- $\tilde{G}(z) = G(z^{-1})$
- $H(z) = H(z^{-1})$

Preview of why to relax orthogonality

FIR

orthogonal

perfect reconstruction

two-channel filter banks

linear phase

only Haar
Biorthogonality conditions

orthogonal case

\[ \langle g_{n-2k}, g_n \rangle_n = \delta_k, \]
\[ \langle h_{n-2k}, h_n \rangle_n = \delta_k, \]
\[ \langle h_{n-2k}, g_n \rangle_n = 0. \]

biorthogonal case

\[ \langle \tilde{g}_{2k-n}, g_n \rangle_n = \delta_k, \]
\[ \langle \tilde{h}_{2k-n}, h_n \rangle_n = \delta_k, \]
\[ \langle \tilde{g}_{2k-n}, h_n \rangle_n = 0, \]
\[ \langle \tilde{h}_{2k-n}, g_n \rangle_n = 0. \]

\[ G(z)\tilde{G}(z) + G(-z)\tilde{G}(-z) = 2 \]
\[ G(z)\tilde{H}(z) + G(-z)\tilde{H}(-z) = \zeta \]
\[ \tilde{G}(z)\tilde{H}(z) + \tilde{G}(-z)\tilde{H}(-z) = \zeta \]
\[ \tilde{H}(z)G(z) + \tilde{H}(z)G(-z) = \zeta \]

After picking allowed \( g, \) the rest follow

After picking allowed \( (g, \tilde{g}) \) pair, the rest follow

Biorthogonal FB: The matching highpass filters

With \( (g, \tilde{g}) \) satisfying

\[ G(z)\tilde{G}(z) + G(-z)\tilde{G}(-z) = 2 \]

pick \( L \) even and

\[ H(z) = -z^{-L+1}\tilde{G}(-z) \]
\[ \tilde{H}(z) = -z^{L-1}G(-z) \]

Verify:

\[ H(z)\tilde{H}(z) + H(-z)\tilde{H}(-z) = 2 \]
\[ \tilde{G}(z)H(z) + \tilde{G}(-z)H(-z) = \zeta \]
\[ \tilde{H}(z)G(z) + \tilde{H}(z)G(-z) = \zeta \]
The biorthogonal lowpass channel

\[ \ell^2(\mathbb{Z}) \xrightarrow{\tilde{g}_n} \downarrow2 \xrightarrow{\alpha} \downarrow2 \xrightarrow{g_n} x_V \]

**Figure 6.12. The biorthogonal lowpass channel.**

- Biorthogonality condition can be expressed as
  \[ G(z)\tilde{G}(z) + G(-z)\tilde{G}(-z) = 2 \]
- Biorthogonality condition can also be expressed as
  \[ D_2\tilde{G}GU_2 = I \]
- Overall operation \( x \rightarrow x_V \) is an **oblique** projection
  \[ P_V = GU_2D_2\tilde{G} \]

Filter bank implementation: Polyphase

\[ \tilde{h}_n \xrightarrow{\downarrow2} \beta \]
\[ \tilde{g}_n \xrightarrow{\downarrow2} \alpha \]

Implementing exactly as drawn is inefficient:
half the computed filter outputs are discarded
Remedy (covered in detail in 6.341):
polyphase implementation
Basic idea of splitting into two polyphase components:
consider odd- and even-indexed samples separately
Using polyphase: One filter

\[ (x \ast \tilde{g})_n = \sum_k \tilde{g}_k x_{n-k} = \sum_{k=2\ell} \tilde{g}_k x_{n-k} + \sum_{k=2\ell+1} \tilde{g}_k x_{n-k} \]

\[ \alpha_n = (x \ast \tilde{g})_{2n} = \sum_{k=2\ell} \tilde{g}_k x_{2n-k} + \sum_{k=2\ell+1} \tilde{g}_k x_{2n-k} \]

even-indexed parts of \( x \) and \( \tilde{g} \)
odd-indexed parts of \( x \) and \( \tilde{g} \)

Polyphase example: Three components

- Split a sequence into 3 by looking at index mod 3
- Alignment/numbering is merely a convention
Polyphase conventions

On the synthesis side we write
\[
G(z) = G_0(z^2) + z^{-1}G_1(z^2)
\]
\[
H(z) = H_0(z^2) + z^{-1}H_1(z^2)
\]

but on the analysis side we write
\[
\tilde{G}(z) = \tilde{G}_0(z^2) + z\tilde{G}_1(z^2)
\]
\[
\tilde{H}(z) = \tilde{H}_0(z^2) + z\tilde{H}_1(z^2)
\]

(Recall: In orthogonal case we had causal synthesis and anticausal analysis.)

Example: lowpass filters (m=2 orthog. design)
Why use both polyphase conventions together?

Use of $z$ and $z^{-1}$ above makes it obvious that the polyphase-domain processing must be an identity for perfect reconstruction (or $z^{-k}I_2$ allowing delay)

- $z$ on analysis side “matches” analysis convention
- $z^{-1}$ on synthesis side “matches” synthesis convention

Deriving polyphase form

\[
\alpha_n = (x * \tilde{g})_{2n} = \sum_{k=2\ell} \tilde{g}_k x_{2n-k} + \sum_{k=2\ell+1} \tilde{g}_k x_{2n-k}
\]

- even-indexed parts of $x$ and $\tilde{g}$
- odd-indexed parts of $x$ and $\tilde{g}$
Two-channel filter bank polyphase representation

\[
\Phi_p(z) = \begin{pmatrix} G_0(z) & H_0(z) \\ G_1(z) & H_1(z) \end{pmatrix}, \quad \Phi_T^T(z) = \begin{pmatrix} \tilde{G}_0(z) & \tilde{G}_1(z) \\ \tilde{H}_0(z) & \tilde{H}_1(z) \end{pmatrix}
\]

Perfect reconstruction depends on the product \( \Phi_p(z) \Phi_T^T(z) \)

What do previously derived/imposed relations between filters imply?

\[
\Phi_p(z) = \begin{pmatrix} G_0(z) & H_0(z) \\ G_1(z) & H_1(z) \end{pmatrix}, \quad \Phi_T^T(z) = \begin{pmatrix} \tilde{G}_0(z) & \tilde{G}_1(z) \\ \tilde{H}_0(z) & \tilde{H}_1(z) \end{pmatrix}
\]

Perfect reconstruction depends on the product \( \Phi_p(z) \Phi_T^T(z) \)

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\]

Perfect reconstruction depends on the product \( \Phi_p(z) \Phi_T^T(z) \)

What do previously derived/imposed relations between filters imply?

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\Phi_p(z) = \begin{pmatrix} G_0(z) & H_0(z) \\ G_1(z) & H_1(z) \end{pmatrix}, \quad \Phi_T^T(z) = \begin{pmatrix} \tilde{G}_0(z) & \tilde{G}_1(z) \\ \tilde{H}_0(z) & \tilde{H}_1(z) \end{pmatrix}
\]

Perfect reconstruction depends on the product \( \Phi_p(z) \Phi_T^T(z) \)

What do previously derived/imposed relations between filters imply?

\[
\Phi_p(z) = \begin{pmatrix} G_0(z) & H_0(z) \\ G_1(z) & H_1(z) \end{pmatrix}, \quad \Phi_T^T(z) = \begin{pmatrix} \tilde{G}_0(z) & \tilde{G}_1(z) \\ \tilde{H}_0(z) & \tilde{H}_1(z) \end{pmatrix}
\]

Perfect reconstruction depends on the product \( \Phi_p(z) \Phi_T^T(z) \)

What do previously derived/imposed relations between filters imply?

\[
\Phi_p(z) = \begin{pmatrix} G_0(z) & H_0(z) \\ G_1(z) & H_1(z) \end{pmatrix}, \quad \Phi_T^T(z) = \begin{pmatrix} \tilde{G}_0(z) & \tilde{G}_1(z) \\ \tilde{H}_0(z) & \tilde{H}_1(z) \end{pmatrix}
\]

Perfect reconstruction depends on the product \( \Phi_p(z) \Phi_T^T(z) \)

What do previously derived/imposed relations between filters imply?
Two-channel filter bank polyphase representation

\[ x \rightarrow \begin{array}{c}
\left( \begin{array}{c}
2 \\
\end{array} \right) \\
\end{array} \xrightarrow{\Phi_p(z)} \Phi_p(z) \xrightarrow{\beta_k} \left( \begin{array}{c}
2 \\
\end{array} \right) \xrightarrow{\Phi^T_p(z)} \tilde{\Phi}_p(z) \xrightarrow{\beta_k} \left( \begin{array}{c}
2 \\
\end{array} \right) \xrightarrow{\tilde{\Phi}_p(z)} \left( \begin{array}{c}
2 \\
\end{array} \right) \xrightarrow{z^{-1}} x
\]

Biorthogonal case:

\[
\tilde{G}(z) = z^{-2\ell-1}H(-z) = \left[ -z^{-2\ell-2}H_1(z^2) \right] + z \left[ -z^{-2\ell-2}H_0(z^2) \right],
\]

\[
\tilde{H}(z) = -z^{-2\ell-1}G(-z) = \left[ z^{-2\ell-2}G_1(z^2) \right] + z \left[ -z^{-2\ell-2}G_0(z^2) \right].
\]

\[
\tilde{\Phi}_p(z) = z^{-\ell-1} \begin{pmatrix} -H_1(z) & G_1(z) \\ H_0(z) & -G_0(z) \end{pmatrix}
\]

\[
\Phi(z)\tilde{\Phi}_p(z) = -z^{-\ell-1} \det \Phi_p(z)I
\]

Perfect reconstruction (possibly with delay) requires ...

Existence of complementary filter

Can I start with any \( g \) and make a biorthogonal filter bank? Almost.

**Proposition 6.5 (Complementary Filters).** Given a causal FIR filter \( G(z) \), there exists a complementary FIR filter \( H(z) \) if and only if the polyphase components of \( G(z) \) are coprime (except for possible zeros at \( z = \infty \)).

Statement itself uses polyphase. Hard to see (very mild) restriction on \( g \) so clearly without polyphase.
Orthogonal + linear phase = Haar (only)

**Proposition 6.7.** There are no two-channel perfect reconstruction, orthogonal filter banks, with filters being FIR, linear phase, and with real coefficients (except for the Haar filters).

Proof in the text uses polyphase and the following:

**Proposition 6.6.** In a two-channel, perfect reconstruction filter bank where all filters are linear phase, the analysis filters have one of the following forms:

1. Both filters are symmetric and of odd lengths, differing by an odd multiple of 2.
2. One filter is symmetric and the other is antisymmetric; both lengths are even, and are equal or differ by an even multiple of 2.
3. One filter is of odd length, the other one of even length; both have all zeros on the unit circle. Either both filters are symmetric, or one is symmetric and the other one is antisymmetric (this is a degenerate case).

Design for annihilation: Recall m=2 orthogonal example

\[ G(z) = (1 + z^{-1})^2 R(z) \text{ with } R(z) \text{ causal,} \]
\[ \text{length 2 (length 1 won't do)} \]

\[ P(z) = (1 + z^{-1})^2 (1 + z)^2 Q(z), \]
\[ \text{with } Q(z) = R(z)R(z^{-1}) (q_n \text{ symmetric}) \]
\[ Q(z) = \frac{1}{16} \left( -z^{-1} + 4 - z \right) \]
\[ \text{satisfies } P(z) + P(-z) = 2 \]

Factor \( Q(z) \propto \left( 1 + \frac{1-\sqrt{3}}{1+\sqrt{3}}z^{-1} \right) \left( 1 + \frac{1-\sqrt{3}}{1+\sqrt{3}}z \right) \]

\[ G(z) \propto (1 + z^{-1})^2 \left( 1 + \frac{1-\sqrt{3}}{1+\sqrt{3}}z^{-1} \right) \]

Biorthogonal: Get to same point and factor \( P(z) \) differently
Design based on polynomial annihilation revisited

Want $P(z) = G(z)\tilde{G}(z)$ and $P(z) + P(-z) = 2$

$$P(z) = (1 + z^{-1})^2 (1 + z) Q(z) \text{ with } Q(z) \propto \left( 1 + \frac{1-\sqrt{3}}{1+\sqrt{3}} z^{-1} \right) \left( 1 + \frac{1-\sqrt{3}}{1+\sqrt{3}} z \right) \text{ works}$$

We have freedom in picking $G(z)$ and $\tilde{G}(z)$

What are the choices?
What are their properties?

Design based on polynomial annihilation revisited

$$G(z) = z^{-1} (1 + z^{-1})^2 (1 + z) = (1 + z^{-1})^3 = 1 + 3z^{-1} + 3z^{-2} + z^{-3},$$
$$\tilde{G}(z) = z(1+z)\frac{1}{4} \left( -\frac{1}{4} z^{-1} + 1 - \frac{1}{4} z \right) = \frac{1}{16} (-1 + 3z + 3z^2 - z^3).$$
$$H(z) = -z^{-3} \frac{1}{16} \left( -1 - 3z + 3z^2 + z^3 \right) = \frac{1}{16} (-1 - 3z^{-1} + 3z^{-2} + z^{-3}),$$
$$\tilde{H}(z) = -z^{3} (1 - 3z^{-1} + 3z^{-2} - z^{-3}) = 1 - 3z + 3z^2 - z^3.$$  
(extra $z^{-1}$ and $z$ factors in lowpass filters and choice of $L = 4$ made synthesis filters causal)

$$G(z) = (1 + z^{-1})^2 (1 + z)^2 = z^{-2} + 4z^{-1} + 6 + 4z + z^2,$$
$$\tilde{G}(z) = \frac{1}{4} \left( -\frac{1}{4} z^{-1} + 1 - \frac{1}{4} z \right) = \frac{1}{16} (-z^{-1} + 4 - z),$$
$$H(z) = -\frac{1}{16} z (z^{-1} + 4 + z),$$
$$\tilde{H}(z) = -z^{-1} (z^{-2} - 4z^{-1} + 6 - 4z + z^2).$$
Design based on finding complementary filter

As long as $G(z)$ has coprime polyphase components, there is an admissible $\tilde{G}(z)$

$$G(z) = \frac{1}{2}z + 1 + \frac{1}{2}z^{-1} = \frac{1}{2}(1 + z)(1 + z^{-1})$$
$$cz^2 + bz + a + bz^{-1} + cz^{-2}$$

$$\begin{bmatrix}
1 & 1/2 & 0 & 0 & 0 \\
0 & 1/2 & 1 & 1/2 & 0 \\
0 & 0 & 0 & 1/2 & 1
\end{bmatrix} \begin{bmatrix}
c \\
b \\
a \\
b \\
c
\end{bmatrix} = \begin{bmatrix}
0 \\
1 \\
0
\end{bmatrix}$$

$$a - 2b + 2c = 0$$

$$\tilde{G}(z) = \frac{1}{8}(-z^2 + 2z + 6 + 2z^{-1} - z^{-2})$$

Iterated filter banks: Main idea

Start with a perfect reconstruction filter bank

What happens if we apply the same analysis/synthesis to $\alpha$? to $\beta$? to both? to $\alpha$ and the new “low-low” signal? . . .