6.342 Lecture 13 — April 1, 2009

Today:
• Constructing wavelet bases for $L^2(\mathbb{R})$ from wavelet bases for $\ell^2(\mathbb{Z})$
• Scaling functions and their properties
• Wavelets and their properties

Homework #4 due Monday, April 6

Midterm Exam Wednesday, April 8, 7:30pm-10:00pm, Room 24-121
• Exam covers through Lecture 14 on April 6
• Lecture 15 on April 8 is not cancelled
• Bring anything provided in class or on the Stellar site (book chapters, papers, HW solutions) and any handwritten notes
• Do not bring other books, calculators, computers, communication devices, etc.
• Fall 2005 and Spring 2007 exams will be available online

Reading:
• Chapter 8 of *The World of Fourier and Wavelets*

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Moving from discrete time to continuous time

We bring from $\ell^2(\mathbb{Z})$ to $L^2(\mathbb{R})$ the following main ideas:

- A filter pair $(g, h)$ such that $(g_n, h_n, g_{-n}, h_{-n})$ gives an orthogonal two-channel filter bank
- The DWT mechanism for iteration

Analogous to the following basis for $\ell^2(\mathbb{Z})$:

$$\left\{h^{(i)}_{n-2^ik}\right\}_{i \in \mathbb{Z}^+, k \in \mathbb{Z}}$$

we get the following basis for $L^2(\mathbb{R})$:

$$\left\{2^{-m/2}\psi(2^{-m}t - n)\right\}_{m \in \mathbb{Z}, n \in \mathbb{Z}}$$
Iterated FB in equivalent single-stage form

Associate a continuous-time function with \( g_n^{(J)} \) and let \( J \to \infty \).

*Daub4*: Length-4 orthogonal \( g \) with two zeros at \( z = -1 \)
Continuous-time functions from iterated g’s

Daub4

Haar: \( g = \frac{1}{\sqrt{2}}[1, 1] \)
Continuous-time functions from iterated g’s

Stretched Haar: \( g = \frac{1}{\sqrt{2}}[1, 0, 0, 1] \)

Definition of scaling function

Define \( \varphi^{(J)}(t) \) from \( g^{(J)}_n \)

Scaling function: If the limit exists, let

\[
\varphi(t) = \lim_{J \to \infty} \varphi^{(J)}(t)
\]

Proposition 8.1 (Necessity of a zero at \( \pi \)). For \( \lim_{J \to \infty} \varphi^{(J)}(t) \) to exist, it is necessary for the filter \( g_n \) to have a zero at \( z = -1 \) or \( \omega = \pi \).
Continuous-time functions from iterated FB

\[ \Phi(J)(\omega) = F[\phi(J)(t)] \]

Figure 8.6. Iteration of a filter without a zero at \( \omega = \pi \). Notice the high frequency oscillations.

Infinite Fourier-domain product

Define \( \varphi^{(J)}(t) \) from \( g_n^{(J)} \)

**Scaling function:** If the limit exists, let

\[ \varphi(t) = \lim_{J \to \infty} \varphi^{(J)}(t) \]

Denote \( \Phi^{(J)}(\omega) = F[\varphi^{(J)}(t)] \)

If the limit exists, let

\[ \Phi(\omega) = \lim_{J \to \infty} \Phi^{(J)}(\omega) \]

In “well-behaved” cases, \( \Phi(\omega) = F[\varphi(t)] \)
Smoothness of scaling functions

\[ G(z) = \left( \frac{1 + z^{-1}}{2} \right)^N R(z) \]

Proposition 8.2 (Smoothness of scaling function). Consider the factorization of the lowpass filter \( G(z) \) in (8.17). Let \( B = \sup_{\omega \in [0, 2\pi]} |R(e^{j\omega})| \). If

\[ B < 2^{N-1/2} \quad (8.22) \]

then the iterated function \( \varphi^{(i)}(t) \) converges pointwise as \( i \to \infty \), to a continuous function \( \varphi(t) \) with Fourier transform

\[ \Phi(\omega) = \prod_{i=1}^{\infty} 2^{-1/2} G(e^{j\omega/2^i}). \]

If we strengthen the bound to

\[ B < 2^{N-k-1/2} \quad \text{for } k \in \mathbb{N} \]

then \( \varphi(t) \) will be continuous and \( k \)-times differentiable (see Exercise 8.3).

Smoothness of scaling functions: Examples

Haar: \( G(z) = \left( \frac{1 + z^{-1}}{2} \right) \sqrt{2} \)

Stretched Haar: \( G(z) = \frac{1}{\sqrt{2}} (1 + z^{-3}) \)

Daubechies length-4:

\[ G(z) = \left( \frac{1 + z^{-1}}{2} \right)^2 \frac{1}{\sqrt{2}} (1 + \sqrt{3} + (1 - \sqrt{3})z^{-1}) \]

Careful: Proposition 8.2 gives a sufficient, but not necessary, condition
Summary of constructions and conditions

- Define piecewise constant $\varphi^{(J)}(t)$ from $g^{(J)}_{n}$
- Define scaling function $\varphi(t) = \lim_{J \to \infty} \varphi^{(J)}(t)$
- Necessary condition for existence of $\varphi(t)$
  Prop. 8.1: Need $G(z)$ to have a zero at $-1$
- Sufficient condition for smoothness of $\varphi(t)$
  Prop. 8.2: In light of Prop. 8.1, write
  $G(z) = 2^{-N}(1 + z^{-1})^{N}R(z)$. Then continuity
  of $\varphi(t)$ ensured by $|R(e^{j\omega})| < 2^{N-1/2}$ for all $\omega$.
- $\Phi(\omega) = \prod_{i=1}^{\infty} 2^{-1/2}G(e^{j\omega/2^{i}})$

Basic properties of scaling function

Orthogonality w.r.t. integer shifts:

$$\langle \varphi(t), \varphi(t - n) \rangle = \delta_{n}$$
Basic properties of scaling function

Orthogonality w.r.t. integer shifts:

\[ \langle \varphi(t), \varphi(t - n) \rangle = \delta_n \]

Basic properties of scaling function

Polynomial reproduction (\(N\) zeros at \(\omega = \pi\)):

\[ \sum_{k=0}^{N-1} a_k t^k \in \text{span} (\{ \varphi(t - n) \}_{n \in \mathbb{Z}}) \]

when restricted to any finite interval

Similar to discrete polynomial reproduction by DWT:

\[ \sum_{k=0}^{N-1} a_k n^k \in \text{span} (\{ g^{(J)}_{n-2^j k} \}_{j,k \in \mathbb{Z}}) \]

(again when restricted to any finite interval)
Polynomial reproduction

Figure 8.9. Reproduction of polynomials by the scaling function and its shifts. Here, \( \varphi(t) \) is based on a 4-tap filter with two zeros at \( \omega = \pi \), and the linear function is reproduced. [Note: Something must be said about the finiteness of a sum \( \gamma \) leading to polynomial reproduction only on an interval.]

Basic properties of scaling function

Two-scale relations:

Because of the infinite product form

\[
\Phi(\omega) = \prod_{i=1}^{\infty} 2^{-1/2} G(e^{j\omega/2^i})
\]

it is also true that

\[
\Phi(\omega) = 2^{-1/2} G(e^{j\omega/2}) \Phi(\omega/2)
\]

By inverse Fourier transform:

\[
\varphi(t) = \sqrt{2} \sum_{n=0}^{L-1} g_n \varphi(2t - n)
\]
Two-scale equation for scaling function

![Figure 8.8. Two-scale equation. The scaling function obtained from the iteration of the 4-tap filter with two zeros at ω = π. It can be written as a linear combination of itself, scaled by a factor of 2, and appropriately shifted and weighted. [This figure is accurate, but it is at some arbitrary scale. It will be redone at scale.]](image)

Definition of wavelet

Define $\psi^{(J)}(t)$ from $h_{n}^{(J)}$ as before

$$\psi^{(J)}(t) = 2^{j/2}h_{n}^{(J)}, \quad \text{for } \frac{n}{2^{J}} \leq t < \frac{n + 1}{2^{J}},$$

Little new to study (iterated part is the same)

Definition 8.3 (Wavelet). Assuming the limit to exist, we define the wavelet in time and frequency domains to be

$$\psi(t) = \lim_{J \to \infty} \psi^{(J)}(t), \quad (8.31)$$

$$\Psi(\omega) = \lim_{J \to \infty} \Psi^{(J)}(\omega). \quad (8.32)$$

In cases of interest, $\Psi(\omega) = \mathcal{F}[\psi(t)]$
Basic properties of wavelet

Because of the infinite product form

\[ \Psi(\omega) = 2^{-1/2} H(e^{j\omega/2}) \prod_{i=2}^{\infty} 2^{-1/2} G(e^{j\omega/2^i}) \]

it is also true that

\[ \Psi(\omega) = 2^{-1/2} H(e^{j\omega/2}) \Phi(\omega/2) \]

By inverse Fourier transform:

\[ \psi(t) = \sqrt{2} \sum_{n=0}^{L-1} h_n \varphi(2t - n) \]

Note: Two-scale relations for wavelet involve scaling function

Two-scale equation for wavelet and scaling fcn

![Wavelet and scaling function graphs](image)

**Figure 8.10.** Wavelet based on 4-tap filter. (a) \( \psi(t) \). (b) The two-scale relation between \( \psi(t) \) and \( \varphi(2t - n) \).
More basic properties of wavelet

Smoothness: Same as smoothness of scaling function

Orthogonality w.r.t. integer shifts:
\[ \langle \psi(t), \psi(t - n) \rangle = \delta_n \]

Zero moments (when \( g \) has \( N \) zeros at \( \omega = \pi \)):
\[ \int_{-\infty}^{\infty} t^n \psi(t) \, dt = 0 \]
for \( n = 0, 1, \ldots, N - 1 \)

Summary of orthonormal wavelet construction

Start with \( g_n \) from an orthonormal PRFB with
\[ G(z) = (1 + z^{-1})^N R(z), \quad \sup_{\omega} |R(e^{j\omega})| < 2^{N-k-1/2} \]

Define \( \varphi^{(J)}(t) \) from \( g^{(J)}_n \)

Scaling function \( \varphi(t) = \lim_{J \to \infty} \varphi^{(J)}(t) \) is continuous and \( k \)-times differentiable

Wavelet \( \psi(t) = \sqrt{2} \sum_{n=0}^{L-1} h_n \varphi(2t - n) \) is nice
Orthonormal wavelet bases

\[ \psi_{m,n}(t) = 2^{-m/2} \psi(2^{-m} t - n) = \frac{1}{2^{m/2}} \psi \left( \frac{t - n 2^m}{2^m} \right) \]

\[ m, n \in \mathbb{Z} \]

Orthogonality w.r.t. integer shifts:

\[ \langle \psi(t), \psi(t - n) \rangle = \delta_n \]

So:

\[ \langle \psi \left( \frac{t}{2^m} \right), \psi \left( \frac{t - n 2^m}{2^m} \right) \rangle = 2^m \delta_n \]

Also:

\[ \langle \psi \left( \frac{t - i 2^k}{2^k} \right), \psi \left( \frac{t - n 2^m}{2^m} \right) \rangle = 0 \]

whenever \( k \neq m \) (regardless of \( i \) and \( n \))
Orthonormal wavelet bases: Completeness

Why can any \( x \in L^2(\mathbb{R}) \) be approximated arbitrarily closely with wavelets?