6.342 Lecture 14 — April 6, 2009

Today:
- Properties of orthonormal wavelet representations
- Multiresolution analysis [lightly, already needed for HW#4]
- Completeness of orthonormal wavelet bases revisited
- Sampling [continued in L15]

Midterm Exam Wednesday, April 8, 7:30pm-10:00pm, Room 24-121
- Exam covers through today’s lecture
- Lecture 15 on April 8 is not cancelled
- Bring anything provided in class or on the Stellar site (book chapters, papers, HW solutions) and any handwritten notes
- Do not bring other books, calculators, computers, communication devices, etc.
- Fall 2005 and Spring 2007 exams will be available online

Reading:
- Chapter 8 of The World of Fourier and Wavelets
- Release 4.0 of The World of Fourier and Wavelets added to Stellar site. In addition to various updates and improvements, some chapters have been reorganized and reordered. Chapter number above is for the version of the book posted 23 Feb 2009.

Review: Scaling function constructions and conditions

- Define piecewise constant \( \varphi^{(J)}(t) \) from \( g_{n}^{(J)} \)
- Define scaling function \( \varphi(t) = \lim_{J \to \infty} \varphi^{(J)}(t) \)
- Necessary condition for existence of \( \varphi(t) \)
  Prop. 8.1: Need \( G(z) \) to have a zero at -1
- Sufficient condition for smoothness of \( \varphi(t) \)
  Prop. 8.2: In light of Prop. 8.1, write
  \[
  G(z) = 2^{-N}(1 + z^{-1})^N R(z). \]
  Then continuity of \( \varphi(t) \) ensured by \( |R(e^{j\omega})| < 2^{N-1/2} \) for all \( \omega \).
- \( \Phi(\omega) = \prod_{i=1}^{\infty} 2^{-1/2} G(e^{j\omega/2^i}) \)
Review: Orthonormal wavelet construction

Start with \( g_n \) from an orthonormal PRFB with
\[
G(z) = \left( \frac{1+z^{-1}}{2} \right)^N R(z), \quad \sup_{\omega} |R(e^{i\omega})| < 2^{N-k-1/2}
\]
Define \( \varphi^{(J)}(t) \) from \( g_n^{(J)} \)

Scaling function \( \varphi(t) = \lim_{J \to \infty} \varphi^{(J)}(t) \) is continuous and \( k \)-times differentiable

Wavelet \( \psi(t) = \sqrt{2} \sum_{n=0}^{2^L-1} h_n \varphi(2t-n) \) is nice

Orthonormal wavelet bases

\[
\psi_{m,n}(t) = 2^{-m/2} \varphi(2^{-m}t-n) = \frac{1}{2^{m/2}} \psi \left( \frac{t-n2^m}{2^m} \right)
\]

Orthogonality w.r.t. integer shifts:

\[
\langle \psi(t), \psi(t-n) \rangle = \delta_n
\]

So by simple change of variable in integral:

\[
\langle 2^{-m/2} \varphi \left( \frac{t}{2^m} \right), 2^{-m/2} \varphi \left( \frac{t-n2^m}{2^m} \right) \rangle = \delta_n
\]

Also, using two-scale relation for \( \psi \) and \( \langle \psi, \phi \rangle = 0 \):

\[
\langle 2^{-k/2} \varphi \left( \frac{t-i2^k}{2^k} \right), 2^{-m/2} \varphi \left( \frac{t-n2^m}{2^m} \right) \rangle = 0
\]

whenever \( k \neq m \) (regardless of \( i \) and \( n \))
Wavelet basis functions in time-frequency plane

Orthonormal wavelet series expansion

\[ x(t) = \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} \beta_n^{(m)} \psi_{m,n}(t) \]

\[ \beta_n^{(m)} = \langle \psi_{m,n}(t), x(t) \rangle, \quad m, n \in \mathbb{Z} \]

Properties:
- Time-frequency localization of basis functions
- Zero moments → polynomial parts give zero \( \beta_n^{(m)} \)'s
- Decay of wavelet series coefficients for smooth functions
- Behavior around singularities
Multiresolution from wavelet bases

\[ \psi_{m,n}(t) = 2^{-m/2} \psi(2^{-m} t - n) = \frac{1}{2^{m/2}} \psi\left(\frac{t - n2^m}{2^m}\right) \quad m, n \in \mathbb{Z} \]

We have that

\[ L^2(\mathbb{R}) = \bigoplus_{m \in \mathbb{Z}} W_m \]

where \( W_m = \text{span}\left(\{\psi_{m,n}(t)\}_{n \in \mathbb{Z}}\right) \), and

\( V_{m-1} = V_m \oplus W_m \) builds up nested spaces

\[ \cdots \subset V_2 \subset V_1 \subset V_0 \subset V_{-1} \subset V_{-2} \]

We got here by defining \( \psi(t) \) through discrete time

Axioms of multiresolution analysis

We impose axioms to have “a multiresolution approximation”:

... \( V_2 \subset V_1 \subset V_0 \subset V_{-1} \subset V_{-2} \) ... embedded approximation spaces

\[ \lim_{m \to -\infty} V_m = \bigcup_{m \in \mathbb{Z}} V_m = L^2(\mathbb{R}) \] “upward” completeness

\[ \bigcap_{m \in \mathbb{Z}} V_m = \{0\} \] “downward” completeness

\[ f(t) \in V_m \Leftrightarrow f(2t) \in V_{m-1} \] scaling

\[ f(t) \in V_m \Leftrightarrow f(t - 2^m n) \in V_m \] shift invariance

\( \{\theta(t-n)\}_{n \in \mathbb{Z}} \) a basis for \( V_0 \) existence of basis

Construct ONB \( \{\varphi(t-n)\}_{n \in \mathbb{Z}} \) from \( \{\theta(t-n)\}_{n \in \mathbb{Z}} \)
Multiresolution analysis and wavelet bases

Multiresolution analysis:
- Axiomatic system that doesn’t require wavelet language
- All continuous time (not starting from filter banks)
- Two-scale relations arise
- Two-scale relations have coefficients
- Can construct discrete-time filters from the coefficients
- We are back to discrete time filter banks!

We start from discrete time because efficient computability and time localization are important to us and obvious in discrete time: Use short filters

Variations on ordinary wavelets

- Base construction on biorthogonal two-channel FB → biorthogonal wavelet bases
- Iterate in other ways (build other trees) → wavelet packet bases
- Start with a PRFB with more than two channels; iterate lowpass channel → bases with non-dyadic multiresolution
- Start with a non-separable multi-dimensional PRFB → nonseparable multidimensional wavelet bases
Nonseparable two-dimensional wavelets
Multiresolution analysis with scaling represented by a matrix $D \in \mathbb{Z}^2$.

Require $D\mathbb{Z}^2 \subset \mathbb{Z}^2$ and $|\lambda_i(D)| > 1$, $i = 1, 2$.

Scaling function satisfies:
$$\varphi(t) = \sqrt{|\det(D)|} \sum_{n \in \mathbb{Z}^2} g_n \varphi(Dt - t)$$

Number of wavelets is number of cosets of $D\mathbb{Z}^2$:
$$|\det(D)| - 1$$

Simplest interesting case: “Twin dragon”

$$D = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad g_n = \begin{cases} 1, & n \in \{(0, 0), (1, 0)\} \\ 0, & \text{otherwise} \end{cases}$$

The scaling function and wavelet both look like this (scaling function positive where shown, wavelet has sign that depends on the color).
Orthonormal wavelet bases: Completeness

Why can any $x \in L^2(\mathbb{R})$ be approximated arbitrarily closely with wavelets?

When can any $x \in L^2(\mathbb{R})$ be approximated arbitrarily closely by samples?

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**Fig. 2.** Schematic representation of the standard three-step sampling paradigm with $T = 1$: 1) the analog input signal is prefILTERED with $h(x)$ (anti-aliasing step), 2) the sampling process yields the sampled representation $c_k(x) = \sum_{k \in \mathbb{Z}} c(k)h(x - k)$, and 3) the reconstructed output $\tilde{f}(x) = \sum_{k \in \mathbb{Z}} c(k)\varphi(x - k)$ is obtained by analog filtering of $c_k$ with $\varphi$. In the traditional approach, the pre- and postfilters are both ideal low-pass: $h(x) = \varphi(x) = \text{sinc}(x)$. In the more modern schemes, the filters can be selected more freely under the constraint that they remain biorthogonal: $\langle \varphi(x - k), \tilde{\varphi}(x - l) \rangle = \delta_{k-l}$.

Unser (2000)
Biorthogonal expansion/reconstruction

Let $H$ be a Hilbert space.
Let $\{\varphi_k\} \subset H$ be a linearly independent set.
Let $\{\tilde{\varphi}_k\} \subset H$ be a set biorthogonal to $\{\varphi_k\}$:

$$\langle \tilde{\varphi}_l, \varphi_j \rangle = \delta_{l-j}$$

Then
- $\{\tilde{\varphi}_k\}$ is a linearly independent set
- $f = \sum_k \langle f, \tilde{\varphi}_k \rangle \varphi_k$ for all $f \in \text{span}\{\varphi_k\}$ with uniqueness of coefficients
- $f = \sum_k \langle f, \varphi_k \rangle \tilde{\varphi}_k$ for all $f \in \text{span}\{\tilde{\varphi}_k\}$ with uniqueness of coefficients

Biorthogonal expansion/reconstruction

Let $\{\varphi_k\} \subset H$ be linearly independent, $\{\tilde{\varphi}_k\}$ and $\{\varphi_k\}$ biorthogonal, $V = \text{span}\{\varphi_k\}$.

$f \in H$ can be written uniquely as

$$f = f_{V^\perp} + \sum_k \langle f, \tilde{\varphi}_k \rangle \varphi_k, \quad \langle f_{V^\perp}, \varphi_i \rangle = 0 \ \forall \ i$$

Having $\tilde{\varphi}_k \in V$ for all $k$ is special: Then

$$\hat{f} = \sum_k \langle f, \tilde{\varphi}_k \rangle \varphi_k$$

is the orthogonal projection of $f$ onto $V$, i.e., $\|f - \hat{f}\|$ is minimum.
Finding a biorthogonal set

To find \( \{\tilde{\varphi}_k\} \subset V \) biorthogonal to \( \{\varphi_k\} \):

Let \( \tilde{\varphi}_k = \sum_i \beta_{k,i} \varphi_i \) for each \( k \).

\[
\delta_{j-k} = \langle \tilde{\varphi}_k, \varphi_j \rangle = \left\langle \sum_i \beta_{k,i} \varphi_i, \varphi_j \right\rangle = \sum_i \beta_{k,i}^* \langle \varphi_i, \varphi_j \rangle
\]

Given \( \{\langle \varphi_i, \varphi_j \rangle\} \), solve linear equations for \( \{\beta_{k,i}\} \)

Important special case:

\( \langle \varphi_i, \varphi_j \rangle \) depends only on \( i-j \)

\( \Rightarrow \beta_{k,i} \) depends only on \( i-k \)

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Shift-invariant expansions for \( L^2(\mathbb{R}) \): Example

Let \( \varphi(t) = \text{sinc}(t) = \frac{\sin(\pi t)}{\pi t} \).

Biorthogonality obtained with \( \tilde{\varphi}(t) = \text{sinc}(t) \).

Interpret

\[
\hat{f}(t) = \sum_k \langle f(t), \tilde{\varphi}(t-k) \rangle \varphi(t-k)
\]
Shift-invariant expansions for $\mathcal{L}^2(\mathbb{R})$: Example

Let $\varphi(t) = \beta^0(t) = \begin{cases} 1, & |t| < \frac{1}{2}; \\ 0, & \text{otherwise}. \end{cases}$

Biorthogonality obtained with $\tilde{\varphi}(t) = \beta^0(t)$.

Interpret

$$\hat{f}(t) = \sum_k \langle f(t), \varphi(t-k) \rangle \varphi(t-k)$$

Sampling when $\langle \varphi(t-i), \tilde{\varphi}(t-j) \rangle \neq \delta_{i-j}$

What part of $f \in \mathcal{L}_2$ (not necessarily in $V(\varphi_1)$ or $V(\varphi_2)$) is observed?

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**Fig. 4.** Sampling for nonideal acquisition devices. The block diagram is essentially the same as in Fig. 2, except for the addition of the digital correction filter $q$. 

Unser (2000)
Geometric interpretation

Impose consistency:

$$\forall f \in H, \left[ \langle f, \varphi_1(t-k) \rangle = \langle \tilde{f}, \varphi_1(t-k) \rangle \quad \forall k \in \mathbb{Z} \right]$$

**Fig. 5.** Principle of an oblique projection onto $V_2$ perpendicular to $V_1$ in the simplified case of one-component signal spaces. (Unser, 2000)

Precise statement

Denote system, with filter $q$, by $P : \mathcal{L}^2(\mathbb{R}) \rightarrow V(\varphi_2)$

"$P$ is a projection onto $V(\varphi_2)$ perpendicular to $V(\varphi_1)$":

$$\forall h \in V(\varphi_2), \quad Ph = h$$
$$\forall g \in \mathcal{L}^2(\mathbb{R}), \quad g - Pg \in V(\varphi_1)$$
$$\forall e \in V(\varphi_1), \quad Pe = 0$$

**Lecture 14**
Uniform (shift-invariant) sampling for $L^2$

We have recovery of $f(t) \in \text{span} \{\{\varphi(t - k)\}_{k \in \mathbb{Z}}\}$ from uniform samples of $f(t) \ast \hat{\varphi}(-t)$

Missing: Adjustable sampling density

With additional condition

$$\sum_{k \in \mathbb{Z}} \varphi(t + k) = 1 \quad \text{for all } t \in \mathbb{R},$$

we can use

$$\hat{f}(t) = \sum_{k} \langle f(t), \hat{\varphi}(t/T - k) \rangle \varphi(t/T - k)$$

to get arbitrarily close to any $f \in L^2(\mathbb{R})$ as $T \to 0$.

Beyond shift-invariant sampling

We have sampling for [notation of Unser (2000)]

$$V(\varphi) = \left\{ f(t) = \sum_{k \in \mathbb{Z}} c_k \varphi(t - k) : c \in \ell^2(\mathbb{Z}) \right\}$$

Alternative:

- Recover $f \in V(\varphi)$ from non-uniform samples
- More complicated but remains linear

Non-subspace sampling [Vetterli, Marziliano, and Blu (2002)]:

$$M(\varphi) = \left\{ f(t) = \sum_{k \in \mathbb{Z}} c_k \varphi(t - t_k) : c \in \ell^2(\mathbb{Z}), t_k \text{s distinct} \right\}$$