6.342 Lecture 17 — April 15, 2009

Today:
• Scheduling project presentations
• Frames (cont.)

Readings:
• Chapter 9 [in most recent numbering] of The World of Fourier and Wavelets
• O. Christensen, An Introduction to Frames and Riesz Bases, 2002.

Project presentations

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Project written report due **11:59pm EDT, Wednesday, May 13**
Frames – outline

- Motivation for sets larger than bases
- Definitions
- Linear analysis and linear synthesis
  - Many solutions give “perfect reconstruction”
  - Projection property from pair of canonical dual frames
- Linear analysis and general synthesis
  - Robustness to additive noise, quantization, erasure
- Fourier frames
- Wavelet frames
- General analysis for linear synthesis
  - Intractability of optimal approximation
  - Greedy algorithm: Matching pursuit
  - Convex relaxation: Basis pursuit

Definitions (and alternatives)

**Frame**: Set \{\varphi_i\}_{i \in I} s.t. for some \(A > 0\) and \(B < \infty\)

\[ A\|x\|^2 \leq \sum_{i \in I} |\langle x, \varphi_i \rangle|^2 \leq B\|x\|^2 \quad \text{for all } x \in H \]

**Lower frame bound**: Any \(A\) above (best \(A\) above)

**Upper frame bound**: Any \(B\) above (best \(B\) above)

**Tight frame**: Frame where one can take \(A = B\)

**Unit-norm frame**: Frame where \(\|\varphi_i\| = 1\) for all \(i \in I\)
(sometimes called “normalized” or “uniform”)

**Parseval tight frame**: Frame where one can take \(A = B = 1\) (sometimes called “normalized”)

Lecture 17
Definitions (and alternatives)

*Frame operator:* $F$ maps $x$ to $\{\langle x, \varphi_i \rangle \}_{i \in I}$

(however, “frame operator” could mean $F^*F$)

*Dual frame:* Any set $\{\tilde{\varphi}_i\}_{i \in I}$ s.t.

$$x = \sum_{i \in I} \langle x, \tilde{\varphi}_i \rangle \varphi_i \quad \text{for all } x \in H$$

*Canonical dual frame:* $\{\tilde{\varphi}_i\}_{i \in I}$ where $\tilde{\varphi}_i = (F^*F)^{-1}\varphi_i$

(however, this could be called simply “the dual frame”)

Canonical dual of tight frame very simple: $\tilde{\varphi}_i = (1/A)\varphi_i$

$$x = \frac{1}{A} \sum_{i \in I} \langle x, \varphi_i \rangle \varphi_i \quad \text{for all } x \in H \text{ (with “best A”)}$$

Robustness to additive noise

$$\hat{x} = F^\dagger \hat{y} = F^\dagger (Fx + \eta) = \hat{F}^* (Fx + \eta) = \sum_{k=1}^M \langle x, \varphi_k \rangle \tilde{\varphi}_k,$$

$$x - \hat{x} = \sum_{k=1}^M \langle x, \varphi_k \rangle \tilde{\varphi}_k - \sum_{k=1}^M \langle x, \varphi_k \rangle \tilde{\varphi}_k = - \sum_{k=1}^M \eta_k \tilde{\varphi}_k.$$

The expected squared-$\ell_2$ error per component (mean-squared error) is

$$\text{MSE} = \frac{1}{M} E \|x - \hat{x}\|^2 = \frac{1}{N} \left[ \sum_{k=1}^M \eta_k \tilde{\varphi}_k \right]^2$$

$$= \frac{1}{N} \left[ \sum_{i=1}^M \sum_{k=1}^M \eta_i \eta_k \tilde{\varphi}_i^* \tilde{\varphi}_k \right] = \frac{1}{N} \sum_{i=1}^M \sum_{k=1}^M \delta_{ik} \sigma^2 \tilde{\varphi}_i^* \tilde{\varphi}_k$$

$$= \frac{1}{N} \sigma^2 \sum_{k=1}^M \|\tilde{\varphi}_k\|^2,$$

Goyal, Kovacevic & Kelner (2001)
\[
\text{MSE} = \frac{1}{N} \sigma^2 \sum_{k=1}^{M} \| \phi_k \|^2 = N^{-1} \sigma^2 \text{tr}(\hat{F} \hat{F}^*) = N^{-1} \sigma^2 \text{tr}((F^*F)^{-1}) \\
= N^{-1} \sigma^2 \text{tr}(V \Lambda^{-1} V^*) = N^{-1} \sigma^2 \text{tr}(\Lambda^{-1}),
\]

where \( F^*F = V \Lambda V^* \) is the spectral decomposition of \( F^*F \). With the \( \{\lambda_i\}_{i=1}^{N} \) denoting the eigenvalues of \( F^*F \), we have

\[
\text{MSE} = \frac{1}{N} \sigma^2 \sum_{i=1}^{N} \frac{1}{\lambda_i},
\]

(19)

**Property 2.2.** For any frame, the sum of the eigenvalues of \( F^*F \) equals the sum of the lengths of the frame vectors. In particular, for a uniform frame the sum of the eigenvalues equals \( N \).

*Proof.* Denote the eigenvalues by \( \{\lambda_i\}_{i=1}^{N} \). Using elementary properties of the trace and the definition of \( F \),

\[
\sum_{i=1}^{N} \lambda_i = \text{tr}(F^*F) = \text{tr}(FF^*) = \sum_{i=1}^{M} \phi_i^* \phi_i = \sum_{i=1}^{M} \| \phi_i \|^2.
\]

Goyal, Kovacevic & Kelner (2001)

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**Robustness to additive noise**

**Theorem 3.1.** When encoding with a uniform frame and decoding with linear reconstruction (16), under the noise model (14), the MSE is minimum if and only if the frame is tight.

**Theorem 3.2.** Consider linear reconstruction (16) with noise \( \eta \) satisfying (14) and define the mean-squared error (MSE) by \( N^{-1} E \| x - \hat{x} \|^2 \). For any frame, the MSE is given by (19) and satisfies

\[
B^{-1} \sigma^2 \leq \text{MSE} \leq A^{-1} \sigma^2.
\]

(20)

For a uniform frame,

\[
\frac{N \sigma^2}{M} \leq \text{MSE} \leq A^{-1} \sigma^2.
\]

(21)

For a uniform tight frame,

\[
\text{MSE} = \frac{N}{M} \sigma^2 = r^{-1} \sigma^2.
\]

(22)

Goyal, Kovacevic & Kelner (2001)
Robustness to erasures

\[ x \in \mathbb{R}^N \rightarrow F \rightarrow y \in \mathbb{R}^M \rightarrow \hat{y} \in \mathbb{R}^M \rightarrow L_E \rightarrow \hat{x} \in \mathbb{R}^N \rightarrow x \in \mathbb{R}^N \]

**FIG. 8.** The full system, as considered in Section 4. A signal expansion is computed with frame operator \( F \). The expansion coefficients are quantized, which is modeled as the addition of a vector \( q \) satisfying (14). The deletion of some quantized expansion coefficients in the transmission is represented by operator \( L_E \). An estimate of the original vector is computed using the linear MMSE estimator.

**Theorem 4.4.** Consider encoding with a uniform frame and decoding with linear reconstruction (16), under noise model (14). The MSE averaged over all possible erasures of one frame element,

\[ \text{MSE}_j = \frac{1}{M} \sum_{k=1}^{M} \text{MSE}_{[k]} . \]

is minimum if and only if the original frame is tight. Also, a tight frame minimizes the maximum distortion caused by one erasure

\[ \max_{k=1,2,...,M} \text{MSE}_{[k]} . \]

Goyal, Kovacevic & Kelner (2001)

Robustness to bounded noise and quantization

\[ x \in \mathbb{R}^N \rightarrow F \rightarrow y \in \mathbb{R}^M \rightarrow Q \rightarrow \hat{y} \in \mathbb{R}^M \rightarrow \text{reconstruction} \rightarrow \hat{x} \in \mathbb{R}^N \]

An estimate \( \hat{x} \) is called **consistent** with \( \hat{y} = Q(Fx) \) when \( Q(F\hat{x}) = \hat{y} \).

The concept of consistency can be used for both deterministic and random quantization.

In a variety of scenarios, consistency reduces MSE from \( O\left(\frac{N}{M}\right) \) to \( O\left(\left(\frac{N}{M}\right)^2\right) \).
Robustness to bounded noise and quantization

Fig. 3. Illustration of consistent reconstruction.

Goyal, Vetterli & Thao (1998)

TABLE I
ALGORITHM FOR CONSISTENT RECONSTRUCTION FROM A QUANTIZED FRAME EXPANSION

1. Form

\[ \bar{F} = \begin{bmatrix} F \\ -F \end{bmatrix} \quad \text{and} \quad \bar{y} = \begin{bmatrix} \frac{1}{2} \Delta + \hat{y} \\ \frac{1}{2} \Delta - \hat{y} \end{bmatrix}. \]

2. Pick an arbitrary cost function \( c \in \mathbb{R}^N \).
3. Use a linear programming method to find \( \hat{x} \) to minimize \( c^T \hat{x} \) subject to \( \bar{F} \hat{x} \leq \bar{y} \).

(Algorithm written for \( Q \) being a quantizer that rounds to nearest multiple of \( \Delta \)).

Goyal, Vetterli & Thao (1998)
The Windowed Fourier Transform

The usual Fourier transform:

\[ F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-j\omega t} \, dt = \langle f(t), e^{j\omega t} \rangle \]

The windowed Fourier transform (WFT):

\[ \text{WFT}_f(\omega, \tau) = \int_{-\infty}^{\infty} g(t-\tau) f(t) e^{j\omega t} \, dt \]

The basis elements

\[ g_{\omega, \tau}(t) = g(t-\tau) e^{j\omega t}, \quad \omega \in \mathbb{R}, \tau \in \mathbb{R} \]

\[ \text{WFT}_f(\omega, \tau) = \langle f(t), g_{\omega, \tau}(t) \rangle \]

Also known as the Gabor transform, or short-time Fourier transform.
Examples of the Window Functions

- the classical choice (Gaussian bell)
  \[ g(t) = \frac{1}{\sqrt{2\pi a}} e^{-t^2/(2a^2)} \]

- sinc function
  \[ g(t) = \sqrt{\alpha} \frac{\sin(\pi t/\alpha)}{\pi t} \]

- raised-cosine window
  \[ g(t) = \begin{cases} \sqrt{\frac{a}{\pi}}(1 + \cos(2\pi t/\alpha)) & t \in [-\alpha/2, \alpha/2] \\ 0 & \text{else} \end{cases} \]

Normalization to ensure unit norm of the window functions

Invertibility, Energy Conservation, and Shift Properties

The inversion formula:
\[ f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} WFT_f(\omega, \tau) g_{\omega,\tau}(t) \, d\omega \, d\tau. \]

Energy conservation:
\[ \|f\|^2 = \int_{-\infty}^{\infty} |f(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |WFT_f(\omega, \tau)|^2 \, d\omega \, d\tau. \]

Spatial shift:
\[ h(t) = f(t - T) \implies WFT_h(\omega, \tau) = e^{-j\omega T} WFT_f(\omega, \tau - T) \]

Frequency shift:
\[ h(t) = e^{j\Omega t} f(t) \implies WFT_h(\omega, \tau) = WFT_f(\omega - \Omega, \tau) \]
Discretization of WFT and the Frame Bounds

WFT: \[ g_{\omega, \tau}(t) = g(t - \tau) e^{j\omega t}, \quad \omega, \tau \in \mathbb{R} \]

Discretization: \[ g_{m,n}(t) = g(t - nt_0) e^{jm\omega_0 t}, \quad t_0 > 0, \omega_0 > 0. \]

Frame bounds: \[ A \leq \frac{2\pi}{\omega_0 t_0} \|g\|^2 \leq B \]

Critical (Nyquist) sampling: \[ \omega_0 t_0 = 2\pi \]

Parameter Choices for WFT Frames

Theorem: (Balian-Low) If \( g_{m,n}(t) = e^{2\pi m t} g(t - n), \ m, n \in \mathbb{Z} \) constitute a frame, then either

\[ \int t^2 |g(t)|^2 \, dt = \infty \]

or

\[ \int \omega^2 |G(\omega)|^2 \, d\omega = \infty. \]
Example: a Gaussian Window

- Window
  \[ g(t) = \pi^{-1/4} e^{-t^2/2} \]

- Sampling
  \[ \omega_0 = f_0 = \sqrt{\lambda} 2\pi \]

- Oversampling factors
  \[ \omega_0 f_0 = 2\pi \lambda \]

\[ \lambda = 0.25, 0.375, 0.5, 0.75, 0.95, 1 \]