6.342 Lecture 20 — April 29, 2009

Today:
• Linear approximation of stochastic sources
• Quantization basics
• Transform coding of Gaussian sources

Readings:
• Draft chapter “Fundamentals of Transform Coding” from a document to be repurposed somehow …

Linear approximation of random sources

Let $x \in \mathbb{R}^N$ be jointly Gaussian with $E[x] = 0$ and $E[xx^T] = R_x$

(Gaussianity is for concreteness. We are using just second-order properties and constraining ourselves to linear approximation.)

How can we best approximate $x$ with $K < N$ terms?
Compression: Bits instead of real coefficients

Basics of scalar quantization

**OED:** To approximate (a signal varying continuously in amplitude) by one whose amplitude is restricted to a prescribed set of discrete values.

Quantizers are most often uniform:
High-resolution analysis of uniform quantizer

\[ D = E[(X - q(X))^2] \]
\[ \approx \sum_{i=1}^{N} \int_{y_i - \Delta/2}^{y_i + \Delta/2} (x - y_i)^2 f(x) \, dx \quad \text{neglecting overload} \]
\[ \approx \sum_{i=1}^{N} f(y_i) \int_{y_i - \Delta/2}^{y_i + \Delta/2} (x - y_i)^2 \, dx \quad \text{by continuity} \]
\[ = \frac{\Delta^2}{12} \sum_{i=1}^{N} f(y_i) \Delta \]
\[ \approx \frac{\Delta^2}{12} \int_{y_N - \Delta/2}^{y_N + \Delta/2} f(x) \, dx \]
\[ \approx \frac{\Delta^2}{12} \quad \text{neglecting overload} \]

If support\( (f(x)) \subset \) limiting granular region,
\[ \lim_{\Delta \to 0} \frac{D}{\Delta^2/12} = 1 \]

High-res. analysis of general scalar quantizer

Consider \( K \)-cell quantization of \( X \in \mathbb{R} \) with smooth pdf \( f_X(x) \)

- optimal quantizer: regular, similarly-sized neighboring cells
- assume overload distortion is negligible
- quantizer described by (normalized) point density \( \lambda(x) \)

\[ \Delta \lambda(x) : \text{approx. fraction of points in } [x - \frac{1}{2} \Delta, x + \frac{1}{2} \Delta] \]

- \( \lambda(x) \) diagram
High-res. analysis of general scalar quantizer

- express MSE using $\lambda$:
  \[ D = \frac{1}{12K^2} \int \frac{f_X(x)}{\lambda^2(x)} \, dx + o(1/K^2) \]

Fixed-rate quantization (minimize MSE given $K$)
- optimally, $\lambda(x) \sim f_X^{1/3}(x)$

Optimization of point density (fixed rate)

Minimizing $D \approx \frac{1}{12} \frac{1}{N^2} \int \frac{f(x)}{\lambda^2(x)} \, dx$:

Hölder’s inequality: When $\frac{1}{a} + \frac{1}{b} = 1$,
\[
\int u(x)v(x) \, dx \leq (u(x)^a \, dx)^{1/a} \left( v(x)^b \, dx \right)^{1/b},
\]
with equality if and only if $u(x)^a = C \cdot v(x)^b$. 


Optimization of point density (fixed rate)

Let \( u(x) = \left( \frac{f(x)}{\lambda^2(x)} \right)^{1/3} \), \( v(x) = \lambda^{2/3}(x) \), 
\( a = 3, \ b = 3/2 \). Then

\[
\int \left( \frac{f(x)}{\lambda^2(x)} \right)^{1/3} \lambda^{2/3}(x) \, dx \leq \left( \int \frac{f(x)}{\lambda^2} \, dx \right)^{1/3} \left( \int \lambda(x) \, dx \right)^{2/3}
\]

Distortion minimized by \( \lambda(x) = \frac{f^{1/3}(x)}{\int f^{1/3}(x') \, dx'} \)

\[
\delta(R) \approx \frac{1}{12} \left( \int f^{1/3}(x) \, dx' \right)^3 2^{-2R}
\]

6\pi\sqrt{3}\sigma^2 \text{ for Gaussian}

High-res. quantization: General variable-rate

With high-resolution approximations, optimal point density is constant.

Uniform quantization gives

\[
H(q(X)) \approx h(X) - \log \Delta,
\]

where \( h(X) = -\int f(x) \log f(x) \, dx \)

\[
\Rightarrow \Delta \approx 2^{h(X)} 2^{-R}
\]

\[
\Rightarrow \delta(R) \approx \frac{1}{12} \frac{2^{2h(X)}}{2\pi \sigma^2} 2^{-2R}
\]

Comparing to fixed-rate quantization,
only space-filling loss
(no oblongitis or point-density loss)
General terminology of source coding

\[ x \xrightarrow{\alpha} \gamma \text{ bits} \xrightarrow{\gamma^{-1}} \beta \xrightarrow{\hat{x}} \]

input \( x \in \text{alphabet } A \subseteq \mathbb{R}^k \)
lossy encoder \( \alpha : A \rightarrow \mathcal{I} \)
reproduction decoder \( \beta : \mathcal{I} \rightarrow \hat{A} \subseteq \mathbb{R}^k \)
lossless encoder \( \gamma : \mathcal{I} \rightarrow \mathcal{J} \)

partition \( \mathcal{S} = \{ S_i = \alpha^{-1}(i) : i \in \mathcal{I} \} \)
(reproduction) codebook \( \mathcal{C} = \{ \beta(i) : i \in \mathcal{I} \} \) of points, codevectors, or reproduction codewords

Performance measures

distortion \( d(x, \hat{x}) = \| x - \hat{x} \|^2 = \sum_{i=1}^{k} |x_i - \hat{x}_i|^2 \)
rate \( R(\alpha, \gamma) = E[r(X)] = \frac{1}{k} E[\ell(\gamma(\alpha(X)))] \)
average distortion \( D(\alpha, \beta) = \frac{1}{k} E[d(X, \beta(\alpha(X))] \)
operational distortion-rate function \( \delta(R) = \inf_{(\alpha, \gamma, \beta) : R(\alpha, \gamma) \leq R} D(\alpha, \beta) \)
operational rate-distortion function \( r(D) = \inf_{(\alpha, \gamma, \beta) : D(\alpha, \gamma) \leq D} R(\alpha, \gamma) \)
operational Lagrangian (weighted r-d) function \( L(\lambda) = \inf_{(\alpha, \gamma, \beta)} D(\alpha, \beta) + \lambda R(\alpha, \gamma) \)
Examples: 2-D uniform source

<table>
<thead>
<tr>
<th>Optimal</th>
<th>Separable</th>
<th>Best separable</th>
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<tbody>
<tr>
<td><img src="image1" alt="Hexagonal Pattern" /></td>
<td><img src="image2" alt="Square Grid" /></td>
<td><img src="image3" alt="Best Separable Grid" /></td>
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</tbody>
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(boundary effects ignored)

Quantizers for 2-D Gaussian source (fixed rate)

\[
E[x] = 0
\]

\[
E[xx^T] = \frac{1}{16} \begin{bmatrix} 13 & 3 & \sqrt{3} \\ 3\sqrt{3} & 7 \\ \end{bmatrix}
\]

30 degree rotation of

\[
E[xx^T] = \begin{bmatrix} 1 & 0 \\ 0 & 1/4 \\ \end{bmatrix}
\]
Effect of dimension

Performance improves with increasing dimension, but so does complexity. Only “space filling loss” is a fundamental limitation of the quantizer dimension. It decreases slowly.

Losses for i.i.d. Gaussian source:

![Graph showing losses for i.i.d. Gaussian source with labels $L_0$ to $L_{1024}$ on the y-axis and dimension $k$ on the x-axis.]

Gray & Neuhoff (1998)

Basic transform coding structure

Transform codes have constrained structure for lower complexity

- $T$ and $U$ are linear operators ($N \times N$ matrices)
- Each $(\alpha_k, \beta_k, \gamma_k)$ is a scalar quantizer

Lower complexity as function of $N$ $\Rightarrow$ Use much larger $N$

Most audio and image compression is with transform codes
Transform codes are constrained quantizers

Standard transform coding results

[Huang & Schultheiss (1963)]

Let \( x \) be jointly Gaussian and assume the \( y_i \)'s are independent and the \( \alpha_i \)'s are optimal fixed-rate quantizers.

Then optimally \( U = T^{-1} \) and \( T \) is orthogonal.
Standard transform coding results

[See, e.g., Gersho & Gray (1992)]

Let $x$ be jointly Gaussian and assume $T^{-1} = T^T = U$. Also assume high-rate approximations of fixed- or variable-rate quantizer performance.

Then $T$ is optimally a KLT ($y_i$'s independent).

More general transform coding result

**Thm. 2 (2000):** Let $T$ be orthogonal and $U = T^{-1} = T^T$. If $D_k = \sigma_k^2 y(R_k)$ for each $k$, a KLT is optimal. (The bit allocation is arbitrary.)

Proof 1: Every Jacobi step in diagonalizing $R_x$ decreases $D$.

Proof 2: Let $(R_1, R_2, \ldots, R_N)$ be any bit allocation. We wish to minimize $D = N^{-1} \sum_{k=1}^N \sigma_k^2 f(R_k)$ by manipulating the $\sigma_k^2$s. The vector $(\sigma_1^2, \sigma_2^2, \ldots, \sigma_N^2)$ lies in the convex polytope defined by the permutations of $\lambda(R_x)$. The minimum of $D$ must be attained at a corner. □
A non-Gaussian approx. ex. [Mallat pp. 387-388]

Consider the random vector $x \in \mathbb{R}^N$ that is a cyclic shift by $k$ of $[1, -1, 0, 0, \ldots, 0]$, with $k$ uniformly distributed on $\{0, 1, \ldots, N-1\}$

How do \{linear, nonlinear\} approximation $L < N$ terms behave?

A similar compression example

Consider the random vector $x \in \mathbb{R}^N$ that has a single $\mathcal{N}(0, 1)$ nonzero component at position $k$, with $k$ uniformly distributed on $\{0, 1, \ldots, N-1\}$

How do \{linear, nonlinear\} (or \{dumb, smart\} or \{fixed, adaptive\}) compression schemes compare?