LECTURE 13

• Readings: Section 6.1

Lecture outline

• Random processes
  • Bernoulli process
    – Definition and basic properties
    – Interarrival times
    – Distribution of kth arrival
    – Merging and splitting

Random processes

• A discrete-time random process is a collection of random variables (defined in the same probabilistic model), e.g.,
  \((X_1, X_2, X_3, \ldots), (X_0, X_1, X_2, \ldots), \text{ or } (\ldots, X_{-1}, X_0, X_1, \ldots)\)

  • New:
    – rarely before had infinite collections
    – interpret the index as time
    – focus is often on dependencies and long-term behavior

  • Third quarter of course:
    – Ch. 6: memoryless processes, discrete and continuous time
    – Ch. 7: Markov processes, discrete time

The Bernoulli process

• A sequence of independent Bernoulli trials \((X_1, X_2, X_3, \ldots)\)

  • At each trial (each \(i \in \{1, 2, \ldots\}\):
    – \(P(\text{success}) = P(X_i = 1) = p\)
    – \(P(\text{failure}) = P(X_i = 0) = 1 - p\)

  • Examples:
    – Sequence of coin tosses
    – Sequence of lottery wins/losses
    – Sequence of ups and downs of the Dow Jones
    – Arrivals of tasks to computer (in time slots)

Basic properties

• \(E[X_i] = \) \(\text{var}(X_i) = \)

  • Let \(S\) be the number of successes/arrivals up to and including time \(n\)
    \(p_S(k) = E[S] = \text{var}(S) = \)

  • Let \(T\) be the number of trials up to and including the first success/arrival
    \(p_T(t) = E[T] = \text{var}(T) = \)

  • \(P(X_i = 1 \text{ for all } i) = \)

Independence and memorylessness

• Trials at disjoint sets of times are independent
  – Example: \(X_4, X_5, X_6, \ldots\) is independent of \(X_1, X_2, X_3\)
  – From any time, the future is independent of the past
  – Any fixed reindexing gives a Bernoulli process, e.g.,
    \((X_4, X_5, X_6, \ldots)\) or \((X_1, X_3, X_5, \ldots)\)

  • Memoryless property
    \(P(T - n = t | T > n) = \)

Arrival and interarrival times

• Let \(Y_k\) be the time of the \(k\)th arrival

  • Increments
    \(T_1 = Y_1, \quad T_k = Y_k - Y_{k-1}, \quad k = 2, 3, \ldots\)
    are called interarrival times
  – Distribution of \(T_1\)
  – Distribution of other \(T_k\)s

• After a rainy day, the number of days until it rains again is geometrically distributed with parameter \(p\), independent of the past. What is the probability that it rains on April 3 and April 7?
Time of the \( k \)-th arrival

- \( Y_k \): number of trials to \( k \)-th success
  - \( \mathbb{E}[Y_k] = \)
  - \( \text{var}(Y_k) = \)
  - \( P(Y_k = t) = \)

Splitting of a Bernoulli process
(\text{using independent coin flips})

\[
\begin{array}{c}
\text{Original process} \\
\text{1 - q}
\end{array}
\]

\[
\begin{array}{c}
\text{Merged process: Bernoulli (p + q - pq)} \\
\text{Bernoulli (p)} \\
\text{Bernoulli (q)}
\end{array}
\]

yields Bernoulli processes

Merging of independent Bernoulli processes

\[
\begin{array}{c}
\text{Bernoulli (p)} \\
\text{Merged process: Bernoulli (p + q - pq)} \\
\text{Bernoulli (p)} \\
\text{Bernoulli (q)}
\end{array}
\]

yields a Bernoulli process

Refining the time scale

\[
\begin{array}{c}
\text{Bernoulli (p)} \\
\text{Merged process: Bernoulli (p + q - pq)} \\
\text{Bernoulli (p)} \\
\text{Bernoulli (q)}
\end{array}
\]

- “Collisions” are a limitation in the model:
  - cannot tell the difference between 1 and 2 arrivals
- Shorter time slots decreases \( p \), can make \( p^2 \ll p \)

Poisson approximation to binomial

- Number of arrivals in \( n \) slots is binomial
  \[
  p_S(k) = \frac{n^k}{(n-k)!} \cdot p^k (1-p)^{n-k}, \quad \text{for } k \geq 0
  \]
- Interesting to think of \( n \to \infty \) with \( \lambda = np \) constant
  \[
  p_S(k) = \frac{n^k}{(n-k)!} \cdot p^k (1-p)^{n-k} = \frac{n^k}{k!} \frac{\lambda^k}{n^k} \left( 1 - \frac{\lambda}{n} \right)^{n-k}
  \]
  \[
  = \frac{n}{n} \frac{n-1}{n} \cdots \frac{n-k+1}{n} \frac{\lambda^k}{k!} \frac{\left( 1 - \frac{\lambda}{n} \right)^{n-k}}{\left( 1 - \frac{\lambda}{n} \right)^{n-k}}
  \]
- For any fixed \( k \geq 0 \), \( \lim_{n \to \infty} (1 - \lambda/n)^{n-k} = e^{-\lambda} \)
  \[
  \lim_{n \to \infty} p_S(k) = \frac{\lambda^k}{k!} e^{-\lambda}
  \]