LECTURE 9

Last time:

- Channel capacity
- Binary symmetric channels
- Erasure channels
- Maximizing capacity

Lecture outline

- Maximizing capacity: Arimoto-Blahut
- Convergence
- Examples
Lemma 1:

\[ I(X; Y) = \max_{\hat{P}_{X|Y}} \sum_{y \in Y} \sum_{x \in X} P_X(x) P_{Y|X}(y|x) \log \left( \frac{\hat{P}_{X|Y}(x|y)}{P_X(x)} \right) \]

Proof:

\[ I(X; Y) = \sum_{x \in X} \sum_{x \in Y} P_{X|Y}(x|y) P_Y(y) \log \left( \frac{P_{X|Y}(x|y)}{P_X(x)} \right) \]

Recall:

\[ P_{X|Y}(x|y) = \frac{P_X(x) P_{Y|X}(y|x)}{\sum_{x' \in X} P_X(x') P_{Y|X}(y|x')} \]

and

\[ P_Y(y) = \sum_{x' \in X} P_X(x') P_{Y|X}(y|x') \]
\[
\begin{align*}
I(X; Y) & - \sum_{y \in Y} \sum_{x \in X} P_X(x) P_{Y|X}(y|x) \\
\log \left( \frac{\hat{P}_{X|Y}(x|y)}{P_X(x)} \right) & = \sum_{y \in Y} \sum_{x \in X} P_Y(y) P_{X|Y}(x|y) \\
\log \left( \frac{\hat{P}_{X|Y}(x|y)}{P_X(x)} \right) & = \sum_{y \in Y} \sum_{x \in X} P_Y(y) P_{X|Y}(x|y) \log \left( \frac{P_{X|Y}(x|y)}{\hat{P}_{X|Y}(x|y)} \right) \\
& \geq \sum_{y \in Y} \sum_{x \in X} P_Y(y) P_{X|Y}(x|y) - \sum_{y \in Y} \sum_{x \in X} P_Y(y) \hat{P}_{X|Y}(x|y) \\
& = 0
\end{align*}
\]
Arimoto-Blahut

Capacity is

\[ C = \max_{P_X} \max_{\hat{P}_{X|Y}} \sum_{y \in Y} \sum_{x \in X} P_X(x) P_{Y|X}(y|x) \]
\[ \log \left( \frac{\hat{P}_{X|Y}(x|y)}{P_X(x)} \right) \]

For fixed \( P_X \), RHS is maximized when

\[ \hat{P}_{X|Y}(x|y) = \frac{P_X(x) P_{Y|X}(y|x)}{\sum_{x' \in X} P_X(x') P_{Y|X}(y|x')} \]

For fixed \( \hat{P}_{X|Y} \), RHS is maximized when

\[ P_X(x) = \frac{e^{\sum_{y \in Y} P_{Y|X}(y|x) \log(\hat{P}_{X|Y}(x|y))}}{\sum_{x' \in X} \left( e^{\sum_{y \in Y} P_{Y|X}(y|x') \log(\hat{P}_{X|Y}(x'|y))} \right)} \]
Arimoto-Blahut

Combining the two means maximization when

\[ P_X(x) = \frac{e^{\sum_{y \in Y} P_{Y|X}(y|x) \log(\hat{P}_{X|Y}(x|y))}}{\sum_{x' \in \mathcal{X}} \left( e^{\sum_{y \in Y} P_{Y|X}(y|x') \log(\hat{P}_{X|Y}(x'|y))} \right)} \]

\[ = \frac{P_X(x) e^{\sum_{y \in Y} P_{Y|X}(y|x) \log \left( \frac{P_{Y|X}(y|x)}{\sum_{x' \in \mathcal{X}} P_X(x') P_{Y|X}(y|x')} \right)}}{\sum_{x' \in \mathcal{X}} P_X(x')} \left( e^{\sum_{y \in Y} P_{Y|X}(y|x') \log \left( \frac{P_{Y|X}(y|x)}{\sum_{x' \in \mathcal{X}} P_X(x') P_{Y|X}(y|x')} \right)} \right) \]

Note also that \( \sum_{x \in \mathcal{X}} P_X(x) = 1. \)

This may be very hard to solve.
Proof:

The first two statements follow immediately from our lemma

For any value of $x$ where $P_{X|Y}(x|y) = 0$, $P_X(x)$ should be set to 0 to obtain the maximum.

To find the maximum over the PMF $P_X$, let us first ignore the constraint of positivity and use a Lagrange multiplier for the $\sum_x P_X(x) = 1$

Then

$$\frac{\partial}{\partial P_X(x)} \left\{ \sum_{x \in X} \sum_{y \in Y} P_X(x) P_{Y|X}(y|x) \log \left( \frac{\hat{P}_{X|Y}(x|y)}{P_X(x)} \right) + \lambda (\sum_{x \in X} P_X(x) - 1) \right\} = 0$$
\[ -\log(P_X(x)) - 1 + \sum_{y \in Y} P_{Y \mid X}(y \mid x) \log \left( \hat{P}_{X \mid Y}(x \mid y) \right) + \lambda = 0 \]

so

\[ P_X(x) = \frac{e^{\sum_{x \in \mathcal{X}} P_{Y \mid X}(y \mid x) \log \left( \hat{P}_{X \mid Y}(x \mid y) \right)}}{\sum_{x \in \mathcal{X}} e^{\sum_{x \in \mathcal{X}} P_{Y \mid X}(y \mid x) \log \left( \hat{P}_{X \mid Y}(x \mid y) \right)}} \]

(this ensures that \( \lambda \) is such that the sum of the \( P_X(x) \)s is 1)

What about the constraint we did not use for positivity?

The solution we found satisfies that.
Convergence of Arimoto-Blahut

Let $P_X^0$ be a PMF and let

$$P_X^{r+1}(x) = P_X^r(x) \frac{c_x \left( P_X^r(x_1), \ldots, P_X^r(x_{|X|}) \right)}{\sum_{x' \in X} c_x \left( P_X^r(x_1), \ldots, P_X^r(x_{|X|}) \right) P_X^r(x')}$$

where

$$c_x \left( P_X^r(x_1), \ldots, P_X^r(x_{|X|}) \right) = e \sum_{y \in Y} P_{Y|X}(y|x) \log \left( \frac{P_{Y|X}(y|x)}{\sum_{x' \in X} P_X(x') P_{Y|X}(y|x')} \right)$$

the sequence $I^r$ of $I(X; Y)$ for $X$ taking the PMF $P_X^R$ for $I^r$ converges to $C$ from below
Convergence of Arimoto-Blahut

Proof:

For any given $P_X^r$, we can increase mutual information by taking

$$P_Y^r|X = \frac{P_X^r(x)P_Y|X(y|x)}{\sum_{x' \in \mathcal{X}} P_X^r(x')P_Y|X(y|x')}$$

With $P_Y^r|X$ fixed, then choose $P_X^{r+1}$ by

$$P_X^{r+1}(x) = \frac{e^{\sum_{y \in \mathcal{Y}} P_Y|X(y|x) \log(P_X^r|Y(x|y))}}{\sum_{x' \in \mathcal{X}} e^{\sum_{y \in \mathcal{Y}} P_Y|X(y|x') \log(P_X^r|Y(x'|y))}}$$

If we define

$$J^r = \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} P_X^{r+1}(x)P_Y|X(y|x) \log \left( \frac{P_X^r|Y(x|y)}{P_X^{r+1}(x)} \right)$$

Then $I^r \leq J^r \leq I^{r+1} \leq J^{r+1} \leq \ldots$

This an upper bounded non-decreasing sequence, therefore it reaches a limit
Convergence of Arimoto-Blahut

Why is the limit $C$?

Let $P_X^*$ be a capacity achieving PMF

$$
\sum_{x \in \mathcal{X}} P_X^*(x) \log \left( \frac{P_X^{r+1}(x)}{P_X^r(x)} \right)
= \sum_{x \in \mathcal{X}} P_X^*(x) \sum_{y \in \mathcal{Y}} P_{Y|X}(y|x)
\log \left( \frac{P_{Y|X}(y|x)}{\sum_{x' \in \mathcal{X}} P_X^r(x') P_{Y|X}(y|x')} \right)
- \sum_{x \in \mathcal{X}} P_X^*(x) \sum_{y \in \mathcal{Y}} P_{Y|X}(y|x)
\log \left( \sum_{x' \in \mathcal{X}} P_X^r(x') \right)
\sum_{y' \in \mathcal{Y}} P_{Y|X}(y'|x') \log \left( \frac{P_{Y|X}(y'|x')}{\sum_{x'' \in \mathcal{X}} P_X^r(x'') P_{Y|X}(y'|x'')} \right)
$$
Convergence of Arimoto-Blahut

By considering the K-L distance, we have that

\[
\sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} P^*_X(x) P_{Y|X}(y|x) \\
\log \left( \frac{\sum_{x' \in \mathcal{X}} P^*_X(x') P_{Y|X}(y|x')}{\sum_{x' \in \mathcal{X}} P^r_X(x') P_{Y|X}(y|x')} \right) \geq 0
\]

so

\[
\sum_{x \in \mathcal{X}} P^*_X(x) \log \left( \frac{P^r+1_X(x)}{P^r_X(x)} \right) \\
\sum_{x \in \mathcal{X}} P^*_X(x) \sum_{y \in \mathcal{Y}} P_{Y|X}(y|x) \\
\log \left( \frac{P_{Y|X}(y|x)}{\sum_{x' \in \mathcal{X}} P^*_X(x') P_{Y|X}(y|x')} \right) \\
- \sum_{x \in \mathcal{X}} P^*_X(x) \sum_{y \in \mathcal{Y}} P_{Y|X}(y|x) \\
\log \left( \sum_{x' \in \mathcal{X}} P^r_X(x') \right) \\
\sum_{y' \in \mathcal{Y}} P_{Y|X}(y'|x') \log \left( \frac{P_{Y|X}(y'|x')}{\sum_{x'' \in \mathcal{X}} P^r_X(x'') P_{Y|X}(y'|x'')} \right)
\]
Convergence of Arimoto-Blahut

Hence

\[ \sum_{x \in \mathcal{X}} P_X^*(x) \log \left( \frac{P_X^{r+1}(x)}{P_X^r(x)} \right) \geq C - J^r \]

Sum over \( r \)

\[ \sum_{r=0}^{m} (C - J^r) \]

\[ \leq \sum_{r=0}^{m} \sum_{x \in \mathcal{X}} P_X^*(x) \log \left( \frac{P_X^{r+1}(x)}{P_X^r(x)} \right) \]

\[ = \sum_{x \in \mathcal{X}} P_X^*(x) \log \left( \frac{P_X^{m+1}(x)}{P_X^0(x)} \right) \]

\[ \leq \sum_{x \in \mathcal{X}} P_X^*(x) \log \left( \frac{P_X^*(x)}{P_X^0(x)} \right) \]

\( C - J^r \geq 0 \) and non increasing, with bounded sum, so it goes to 0, hence \( J^r \) converges to \( C \)

In practice, convergence can be very slow
Example
Other types of maximization

Interior point methods

Cutting plane algorithms