LECTURE 10

- **Readings:** Section 3.6; start Section 4.1

Lecture outline

- Review
- Continuous Bayes’ rule
- Derived distributions
Review (from L07, emphasis added)

- Multiplication rule for **discrete** random variables:
  \[ p_{X,Y}(x,y) = p_Y(y) p_{X|Y}(x | y) \quad \text{and} \quad p_{X,Y}(x,y) = p_X(x) p_{Y|X}(y | x) \]

- Bayes’ rule for **discrete** random variables:
  \[ p_{X|Y}(x | y) = \frac{p_X(x) p_{Y|X}(y | x)}{p_Y(y)} \]

(where the conditional PMF is defined)
Continuous counterparts

- Multiplication rule for continuous random variables:

\[ f_{X,Y}(x,y) = f_Y(y) f_{X|Y}(x \mid y) \quad \text{and} \quad f_{X,Y}(x,y) = f_X(x) f_{Y|X}(y \mid x) \]

- Bayes’ rule for continuous random variables:

\[ f_{X|Y}(x \mid y) = \frac{f_X(x) f_{Y|X}(y \mid x)}{f_Y(y)} \]

(where the conditional PDF is defined)
Example: Two light bulbs

- Suppose light bulbs have lifetimes that are independent and identically exponentially distributed.
- One is installed at noon, burns out, and is replaced immediately. The replacement burns out at 2pm.
- What is the distribution of the time at which the first bulb burns out?
Conditioning an event on a continuous random variable

• Suppose $f_{Y|A}(y)$ and $f_{Y|A^c}(y)$ are known

• Defining $P(A \mid Y = y)$ requires care because $P(\{Y = y\}) = 0$
  – When $\delta > 0$ is very small and $f_Y(y) > 0$,
    
    $$P(A \mid Y = y) \approx P(A \mid \{Y \in [y, y + \delta]\})$$

    $$= \frac{P(A)P(\{Y \in [y, y + \delta]\} \mid A)}{P(\{Y \in [y, y + \delta]\})}$$

    $$\approx \frac{P(A)f_{Y|A}(y)\delta}{f_Y(y)\delta}$$

    $$= \frac{P(A)f_{Y|A}(y)}{f_Y(y)}$$

    $$= \frac{P(A)f_{Y|A}(y)}{P(A)f_{Y|A}(y) + P(A^c)f_{Y|A^c}(y)}$$
Discrete $X$, Continuous $Y$

\[
P(\{X = x\} \mid \{Y = y\}) = \frac{p_X(x) f_{Y\mid X}(y \mid x)}{f_Y(y)}
\]

\[
p_{X\mid Y}(x \mid y) = \frac{p_X(x) f_{Y\mid X}(y \mid x)}{f_Y(y)}
\]

\[
f_Y(y) = \sum_x p_X(x) f_{Y\mid X}(y \mid x)
\]

Example:

- $X$: a discrete signal; “prior” $p_X(x)$
- $Y$: noisy version of $X$
- $f_{Y\mid X}(y \mid x)$: continuous noise model
Continuous $X$, Discrete $Y$

\[
f_{X|Y}(x \mid y) = \frac{f_X(x)p_{Y|X}(y \mid x)}{p_Y(y)}
\]

\[
p_Y(y) = \int_x f_X(x)p_{Y|X}(y \mid x) \, dx
\]

Example:

- $X$: a continuous signal; “prior” $f_X(x)$
  (e.g., intensity of light beam)
- $Y$: discrete r.v. affected by $X$
  (e.g., photon count)
- $p_{Y|X}(y \mid x)$: model of the discrete r.v.
  (e.g., Poisson with parameter that depends on $x$)
Derived distributions

- When $Y = g(X)$ and the distribution of $X$ is known, the distribution of $Y$ is **derived** from the distribution of $X$.

- Term and techniques apply to functions of any number of variables $g(X, Y, Z)$, etc.

When not to find them

- Don’t need distribution of $g(X)$ to compute $E[g(X)]$:
  \[
  E[g(X)] = \int g(x)f_X(x) \, dx \quad \text{(continuous case)}
  \]
  \[
  E[g(X)] = \sum g(x)p_X(x) \quad \text{(discrete case)}
  \]
Finding derived distributions: Discrete case

• Obtain probability mass for each possible value of $Y = g(X)$:

\[ p_Y(y) = P\{g(X) = y\} = \sum_{x: g(x) = y} p_X(x) \]
Finding derived distributions: Continuous case

- Two-step procedure:
  - Get CDF of $Y$: $F_Y(y) = P(Y \leq y)$
  - Differentiate to get
    
    $$f_Y(y) = \frac{dF_Y}{dy}(y)$$

- Example:
  - $X$: uniform on $[0,2]$
  - Find PDF of $Y = X^3$
The PDF of $Y = aX + b$

$Y = 2X + 5$:

$$f_Y(y) = \frac{1}{|a|} f_X \left( \frac{y - b}{a} \right)$$

- Check: if $X$ is normal, then $Y = aX + b$ is also normal.
Preview: A more general formula

- Consider $Y = g(X)$, where $g$ is strictly monotonic.

- Event $x \leq X \leq x + \delta$ is the same as $g(x) \leq Y \leq g(x + \delta)$

- Approximately:

$$g(x) \leq Y \leq g(x) + \delta \left| \frac{dg}{dx}(x) \right|$$