LECTURE 14

• **Readings:** Start Section 6.2

Lecture outline

• Review of Bernoulli process
• Definition of Poisson process
• Distribution of number of arrivals
• Distribution of interarrival times
• Other properties of the Poisson process
Review: Bernoulli process

- $X_1, X_2, \ldots$: independent Bernoulli RVs with success prob. $p$
- Number of arrivals in $n$ time slots: binomial PMF
- Interarrival times: independent with geometric PMF
- Time of $k$th arrival: Pascal PMF of order $k$
- Independence and memorylessness
Limitation of the Bernoulli arrival model

Consider cars entering Stata Center parking garage
- Mark each 10-second interval with an entry a “success”
- Good model? (captures what is happening?)

... entering any parking garage in the world
- Mark each 10-second interval with an entry a “success”
- Good model? (captures what is happening?)
Definition of the Poisson process

- Defining characteristics:
  - **Time homogeneity**: Probability of \( k \) arrivals in an interval of duration \( \tau \) is some function \( P(k, \tau) \)
  - **Independence**: Numbers of arrivals in disjoint time intervals are independent
  - **Small interval probabilities**: For very small \( \delta \),

\[
P(k, \delta) \approx \begin{cases} 
1 - \lambda \delta, & \text{if } k = 0; \\
\lambda \delta, & \text{if } k = 1; \\
0, & \text{if } k > 1,
\end{cases}
\]

where \( \lambda \) is called the **rate**
PMF of number of arrivals $N_t$

- Finely discretizing $[0, t]$, process is approximately Bernoulli
- $N_t$ (of discrete approximation) is binomial
- Taking $\delta \to 0^+$ (or $n \to \infty$) allows approximation from L13:
  \[
P(k, t) = \frac{(\lambda t)^k e^{-\lambda t}}{k!}, \quad k = 0, 1, \ldots
  \]
  - (consider this exact)
- $\mathbb{E}[N_t] = \lambda t$, $\text{var}(N_t) = \lambda t$
Examples

Email arrives as Poisson process with rate $\lambda = 0.4$ per hour.

Recall general form $P(k, t) = \frac{(\lambda t)^k e^{-\lambda t}}{k!}, \ k = 0, 1, \ldots$

- $P(0 \text{ new msgs between noon and 12:30pm}) =$
- $P(1 \text{ new msg between noon and 12:30pm}) =$

- number of msgs between noon and 1pm $\sim$
- number of msgs between 1pm and 3pm $\sim$
- number of msgs between noon and 3pm $\sim$
Time of first arrival \( T \)

\[
F_T(t) =
\]

\[
f_T(t) =
\]

\[
P(T > t + s | T > t) =
\]
Time of $k$th arrival $Y_k$

- All interarrival times independent with exponential distribution: $f_T(t) = \lambda e^{-\lambda t}, \ t \geq 0$

- $Y_k = T_1 + T_2 + \cdots + T_k$

- Erlang PDF of order $k$: $f_{Y_k}(y) = \frac{\lambda^k y^{k-1} e^{-\lambda y}}{(k-1)!}, \ y \geq 0$
### Bernoulli/Poisson correspondences

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<th>Poisson</th>
<th>Bernoulli</th>
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<td>times of arrival</td>
<td>continuous</td>
<td>discrete</td>
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<tr>
<td>arrival rate</td>
<td>$\lambda$ per unit time</td>
<td>$p$ per trial</td>
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<td>number of arrivals</td>
<td>Poisson PMF</td>
<td>binomial PMF</td>
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<td>interarrival times</td>
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<td>$k$th arrival time</td>
<td>Erlang PDF of order $k$</td>
<td>Pascal PMF of order $k$</td>
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Adding (merging) Poisson processes

- Merging independent Poisson processes gives a Poisson process

What is the probability that the next arrival comes from the first process?

- Sum of independent Poisson random variables is Poisson